Chapter 7.2: Graphs of Polar Equations

The graph of a polar equation of the form $r = f(\theta)$ consists of all points of the form (r, θ) whose coordinates satisfy the equation. Many curves, especially more complex curves, are more easily express as a polar, rather than a rectangular equation. Although we can certainly graph polar curves in the polar plane, it is still helpful to refer to our old rectangular grid for analysis.

Rectangular (Cartesian) Grid	**Polar Grid**
origin	pole
positive x-axis	polar axis
y-axis	line $\theta = \frac{\pi}{2}$

Example 1:
Sketch the graph of the polar equation $r = 4$, then convert it to an equivalent rectangular equation.

Example 2:
Sketch the graph of the polar equation $\theta = -\frac{4\pi}{3}$, then convert it to an equivalent rectangular equation.

Example 3:
Sketch the graph of the polar equation $r = 3\cos\theta$. Predict what the graph of $r = 5\cos\theta$, $r = 8\cos\theta$, and $r = -4\cos\theta$ will look like. Can you generalize about polar equations of the form $r = 2a\cos\theta$, and $r = 2a\sin\theta$ for $a \neq 0$. What will $a > 0$ do to the graph? What will $a < 0$ do to the graph. What do you notice about any symmetries?

Graphs of the form $r = 2a\cos\theta$ or $r = 2a\sin\theta$

Graphs of the form $r = 2a\cos\theta$ or $r = 2a\sin\theta$ ($a \neq 0$) will be **circles** centered OFF the pole with radius of $|a|$, traced out once counterclockwise from $0 \le \theta \le \pi$.

- Graphs involving cosine will have polar axis (*x*-axis) symmetry
- Graphs involving sine will have symmetry along the line $\theta = \frac{\pi}{2}$ (*y*-axis)
- If $a > 0$, the graph will live in "positive" land. If $a < 0$, the graph will live in "negative" land.

Example 4:
Sketch the graph of $r = 3 - 3\sin\theta$. Can you predict what the graph of $r = 4 + 4\sin\theta$, $r = 2 + 2\cos\theta$, $r = 2 - 2\cos\theta$, and $r = -2 - 2\cos\theta$ will look like? What do you notice about any symmetries?

Graphs of the form $r = a \pm a\cos\theta$ and $r = a \pm a\sin\theta$

Graphs of the form $r = a \pm a\cos\theta$ and $r = a \pm a\sin\theta$ ($a \neq 0$) will be **cardioids**, traced out once counterclockwise from $0 \leq \theta \leq 2\pi$.

- Graphs involving cosine will have polar axis (*x*-axis) symmetry
- Graphs involving sine will have symmetry along the line $\theta = \frac{\pi}{2}$ (*y*-axis)

Example 5:
Sketch the graphs of $r = 3 - 4\sin\theta$, $r = 3 - 5\sin\theta$, $r = 2 - 5\sin\theta$. What do you predict the graphs of $r = 3 - 4\cos\theta$, $r = 3 - 5\cos\theta$, $r = 2 - 5\cos\theta$ will look like? How about $r = 3 + 4\sin\theta$ and $r = -3 + 4\sin\theta$? What do you notice about any symmetries?

Graphs of the form $r = a + b\cos\theta$ and $r = a + b\sin\theta$, $b > a$

Graphs of the form $r = a + b\cos\theta$ and $r = a + b\sin\theta$, $|b| > |a|$ ($b, a \neq 0$) will be **limaçons**, traced out once counterclockwise from $0 \leq \theta \leq 2\pi$.

- Graphs involving cosine will have polar axis (*x*-axis) symmetry
- Graphs involving sine will have symmetry along the line $\theta = \frac{\pi}{2}$ (*y*-axis)
- The diameter of the outer loop will be $|b| + |a|$
- The diameter of the inner loop will be $|b| - |a|$

The values of θ for which $r = 0$ are called the **polar zeros** of the function $r = f(\theta)$. Knowing the angles where these occur is very important! You can find these prior to sketching a polar curve by setting the polar equation, solved for *r*, equal to zero, then solve using the Unit Circle or a calculator.

Note: In the above form, if $|a| > |b|$, the graphs will have no polar zeros, and the graph will resemble a circle flattened on one end. These will be called **dimpled or convex limaçons**. Cardioids, limaçons, and dimpled/convex limaçons all belong to the same family of polar curves.

Example 6:
Identify each of the following polar curves, then find the polar zeros of each of them on $0 \le \theta \le 2\pi$.
(a) $r = 5 - 5\sin\theta$ (b) $r = 4\cos\theta$ (c) $r = -6 - 3\cos\theta$ (d) $r = -2 + 4\sin\theta$

Example 7:
Find the polar zeros on $0 \le \theta \le 2\pi$, then sketch the graph of the polar curve $r = 3\sin 2\theta$. Use your **calculator** to then explore what the graphs of $r = 5\sin 2\theta$, $r = 5\sin 4\theta$, $r = 3\cos 2\theta$, $r = 4\cos 6\theta$ will look like. What do you notice about any symmetries?

Example 8:
Find the polar zeros on $0 \le \theta \le 2\pi$, then sketch the graph of the polar curve $r = 3\sin 3\theta$. Use your **calculator** to then explore what the graphs of $r = 5\sin 3\theta$, $r = 5\sin 5\theta$, $r = 3\cos 3\theta$, $r = 4\cos 7\theta$ will look like. What do you notice about any symmetries?

Graphs of the form $r = a\cos(n\theta)$ and $r = a\sin(n\theta)$

Graphs of the form $r = a\cos(n\theta)$ and $r = a\sin(n\theta)$ ($a, n \neq 0$) will be **rose curves.**

- Graphs involving cosine will have polar axis (x-axis) symmetry
- Graphs involving sine will have symmetry along the line $\theta = \frac{\pi}{2}$ (y-axis)
- If n is even, the rose curve will have $2n$ petals of length a, traced out once from $0 \leq \theta \leq 2\pi$.
- If n is odd, the rose curve will have n petals of length a, traced out once from $0 \leq \theta \leq \pi$.

As you have seen, many polar curves exhibit certain symmetries. If we know the symmetry in advance, we can use it to assist us in sketching it. There are three types of symmetries.

Tests for Symmetry

1. If a polar equation is unchanged when we replace θ by $-\theta$, then the graph is symmetric about the polar axis (x-axis symmetry).
2. If the equation is unchanged when we replace r by $-r$, then the graph is symmetric about the pole (origin).
3. If the equation is unchanged when we replace θ by $\pi - \theta$, the graph is symmetric about the vertical line $\theta = \frac{\pi}{2}$ (y-axis).

Example 9:

For the polar equation $r^2 = 4\cos 2\theta$, determine any of the above symmetries, then sketch a graph of the polar curve. Determine an equivalent equation in rectangular form.

Example 10:

Using your graphing calculator in an appropriate window, sketch the entire graph of $r = \cos\left(\frac{2\theta}{3}\right)$. Sketch the graph.

Some Common Polar Curves