Chapter 8.2: The Integral

You can think of Calculus as a double-wide trailer. In one "width" of it lives differential calculus. In the other half lives what is called **integral calculus**. We have already explored a few rooms in the differential part of the house which contained the mathematics for describing how things change or grow at a specific point or moment in time, i.e. **slope**. In integral calculus, we're interested in finding things like **areas** and volumes of irregular objects. These two seemingly completely different ideas, slopes and areas, are connected by the same idea of the limit, which you'll explore more next year in a calculus course.


Ich bin Happy.

It is the nature of mathematics to use things for which we have a keen and thorough understanding to help us explain and develop tools for things we don't yet fully understand. Just as we used lines to help us talk about how curves change in differential calculus, similarly with integral calculus, we will use the areas of a simple object like a rectangle to help us find areas of irregular regions. This method is of partitioning a region into rectangles and adding up all the areas is called the method of **Riemann Sums**, after German Mathematician Georg Friedrich Bernhard Riemann.

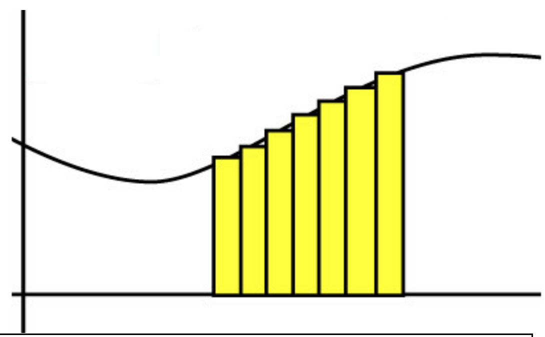
Here's the Idea:

Example 1:
Identify the region bounded by the function $f(x)=x$, the *x*-axis, and the vertical line $x=4$. Partition this region into 4 equal widths. By using both the left and right endpoints of each interval, draw 4 rectangles of equal width. In each case, find the sum of the areas of the 4 rectangles to approximate the area of the region. Compare this with the actual area. How can we make the approximation of rectangular areas get closer to the actual area??

Example 2:

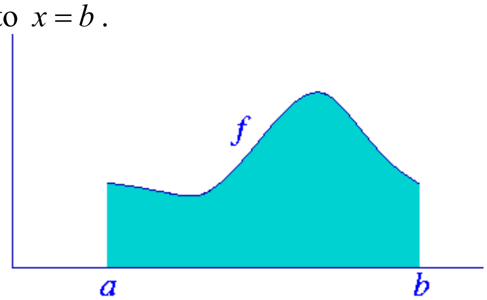
Find the area of the region bounded by $f(x)=x^2$, the x-axis, and the vertical lines $x=0$ and $x=2$, using four subintervals of equal width using both Left and Right endpoint Riemann Sums. Using this Java Applet, determine the limit of the upper and lower sums as $\Delta x \to 0$ and $n \to \infty$.

We can avoid this finite, numeric approximation process by using the **Fundamental Theorem of Calculus**. Not only will this theorem allow us to compute the area by a single calculation, but it will give us the **exact** area as well, not an approximation. Here's the essence of the Fundamental Theorem of Calculus.



The Fundamental Theorem of Calculus (FTOC)

Suppose we have a function $y=f(x)$. We are interested in finding the area under the curve of $y=f(x)$, bounded by the x-axis from $x=a$ to $x=b$.



1. Find a function $F(x)$ such that $F'(x)=f(x)$. $F(x)$ is called an **antiderivative** of $f(x)$.
2. Then the area of the region is simply $F(b)-F(a)$.

Amazing, mathemagical result, eh?

Example 3:
Find the area of the region from Examples 1 and 2using the FTOC.

The process in Example 3 has a precise mathematical notation. We can rewrite the instructions from Example 3 in the following way:

$$\int_0^4 x\,dx = \quad \text{and} \quad \int_0^2 x^2\,dx =$$

We read this first one as ***"the integral of x from zero to four with respect to x."*** The integral symbol, not surprisingly, resembles the letter "S" for . . . Sum! . . . the sum of infinitely many areas of infinitely many rectangles. It looks like this mathematically:

$$\lim_{\Delta x \to 0} \sum_a^b f(x)\Delta x = \int_a^b f(x)\,dx$$

The right side of this equation is known as a **definite integral**, which gives a definite answer which, in the examples we're working, represents the area of the region bounded by the graph of $f(x)$, the x-axis from $x = a$ to $x = b$.

There is an intermediate step involved when using this new, concise notation.

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

"When" bar, or "Evaluation" bar

Example 4:

Evaluate $\int_0^{\pi} \sin x dx$, then illustrate what this represents geometrically.

Example 5:

Evaluate $\int_0^{\ln 4} e^x dx$, then illustrate what this represents geometrically.

Example 6:

Evaluate $\int_a^{2a} \frac{1}{x} dx$ for $a > 0$, then illustrate what this represents geometrically.

So far we have limited our study of area to regions ABOVE the *x*-axis. These have been positive numbers (as area CANNOT be negative, right?). The process in the above examples using the integral symbol is called, in general, **integration**. When integrating from left to right, regions BELOW the *x*-axis are counted **negatively**.

Example 7:

Evaluate (a) $\int_0^{2\pi} \sin x dx$, (b) $\int_0^{3\pi/2} \sin x dx$, and (c) $\int_0^{5\pi/2} \sin x dx$, then illustrate what these represent geometrically.

To understand why and how the FTOC works, you'll have to wait until next year (or you can ask Khan or Google it.) Here, though, is an important piece to understanding why it works.

Question: If $f'(x) = g'(x)$, does it necessarily follow that $f(x) = g(x)$?

Example 8:
List several functions whose derivative is $3x^2$.

**In general, if $F'(x) = f(x)$, then $\int f(x)dx = F(x) + C$, where C is some constant. That is, if two functions have the same derivative, then those two functions differ only by a constant C.

Note 1: $\int f(x)dx$ is known as an indefinite integral, since its answer is an indefinite, variable answer.
Note 2: $F(x) + C$ is known as the general antiderivative, as opposed to **an** antiderivative.

Example 9:

Evaluate (a) $\int 3x^4 dx$ and (b) $\int \frac{2}{3}\sqrt{x}dx$

Trying to find antiderivatives intuitively can sometimes be an arduous task. Luckily, we can use our knowledge of finding derivatives using the power rule (multiply, then subtract) in reverse order using inverse operations (add, then divide) to find antiderivatives.

The Power Rule for antidifferentiation (or integration)

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C$$

As it was for differentiation, similar rule apply for integration

- rewriting prior to using the rule is often the key.
- The integral of the sum is the sum of the integrals

Example 10:

Evaluate (a) $\int 3x^7 dx$ (b) $\int 5\sqrt[4]{x^3}dx$ (c) $\int\left(\frac{3}{\sqrt{x}} - 5\sin x + 3x^{-1}\right)dx$

Example 11:

Evaluate (a) $\int 2x(x-3)^2 dx$ (b) $\int\left(\frac{2x^2 - 3\sqrt[3]{x} - 4 + x\cos x}{x} + e^x\right)dx$