

AP Statistics, NOTES:  $t$ -distributions.

If we don't know the true population standard deviation  $\sigma$  (remember it is unreasonable to assume we WILL know it), we can use the sample standard deviation,  $s$ , to approximate  $\sigma$ . Our accuracy depends on the sample size of course.

Definition: The **Standard Error** (SE) is the estimated standard deviation from data size  $n$ .

$$SE = \frac{s}{\sqrt{n}}$$

As  $n$  increases, SE approaches  $\sigma$ .

\*\*When we calculate a test statistic using SE instead of  $\sigma$ , the result is NOT a normal distribution. It forms a new distribution called a  $t$ -distribution.

Our new test statistic is no longer  $z$  but rather called the **one-sample  $t$  statistic**, and it is calculated in much the same way.

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{\bar{x} - \mu}{SE}$$

For each value of  $n$ , the distribution is different. As  $n$  increases, the  $t$ -distribution approaches the normal distribution. We say a  $t$ -distribution of size  $n$  has  $n - 1$  **degrees of freedom**, df.

What does this  $t$  stat tell us? Well, the same as ANY standardized test statistic: how far  $\bar{x}$  is from its mean  $\mu$  in standard deviation units.

What does the standard normal  $t$ -distribution look like compared with the  $z$ -distribution (Normal curve)? Good question. Both are symmetric with respect to the center 0, but the  $t$ -distribution is shorter and has more area/probability in the tails (since we're assuming more variability by using  $s$  instead of  $\sigma$ ).

The greater the sample size, the greater the degrees of freedom, the less variability, and the closer the  $t$ -distribution becomes normal.

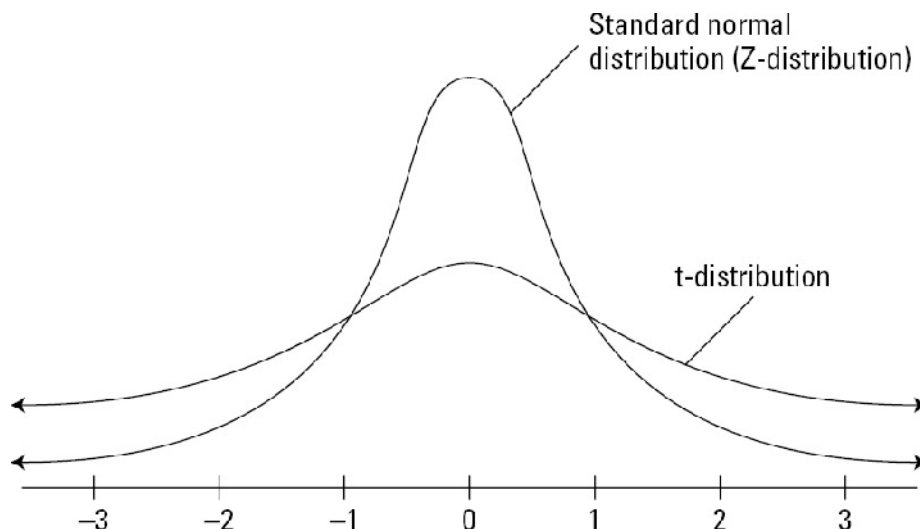


Table C gives the area ABOVE the critical  $t$ -values for varying degrees of freedom. Notice in each of the columns how the numbers approach the values in the last rows (which are the  $z$ -values).

Of course, we have to test our assumptions in order to use our inference methods.

- The data is from a simple random sample from the population of interest.
- Observations from the population must be normal or approximately normal. The Central Limit Theorem tells us that if  $n > 30$ , the distribution of  $\bar{x}$  will be approximately normal.
- The population must be at least 10 times the sample size.

\*\*If you are given data, you can determine whether the distribution is approximately normal by doing

- a stemplot and looking to see if the distribution is symmetrical and single peaked. If it's not, proceed with caution.
- b. Sketch a normal probability plot (the last option on your calculator's scatterplot menu) and look to see if the graph is roughly linear.

The one-sample  $t$  procedures are similar in both reasoning and computational detail to the  $z$  procedures of Chapter 10.

**Example:**

Environmentalists, government officials, and vehicle manufacturers are all interested in studying the auto exhaust emissions produced by motor vehicles. The major pollutants in auto exhaust are hydrocarbons, monoxide, and nitrogen oxide (NOX). The following table gives the NOX levels in grams/mile for a sample of light-duty engines of the same type.

**Amount of Nitrogen Oxides emitted by light-duty engines (grams/mile)**

1.28	1.17	1.16	1.08	0.60	1.32	1.24	0.71	0.49	1.38	1.20	0.78
0.95	2.20	1.78	1.83	1.26	1.73	1.31	1.80	1.15	0.97	1.12	0.72
1.31	1.45	1.22	1.32	1.47	1.44	0.51	1.49	1.33	0.86	0.57	1.79
2.27	1.87	2.94	1.16	1.45	1.51	1.47	1.06	2.01	1.39		

- a. Construct a 95% confidence interval for the mean amount of NOX emitted by light-duty engines of this type.
- b. If the cars can only pass inspection if the NOX level is less than or equal to 1.2 grams per mile, is the data significant to suggest that this batch of engines exceeds the requirement? Compare at both a 5% and 1% level.

*Step 1: Identify the population of interest and the parameter you want to draw a conclusion about.*

The population of interest is ALL light-duty engines of this type. We want to estimate  $\mu$ , the mean amount of the pollutant NOX emitted, for all of these engines.

*Step 2: Choose the appropriate inference procedure.*

Since we do not know  $\sigma$ , we should use the one-sample  $t$  procedures.

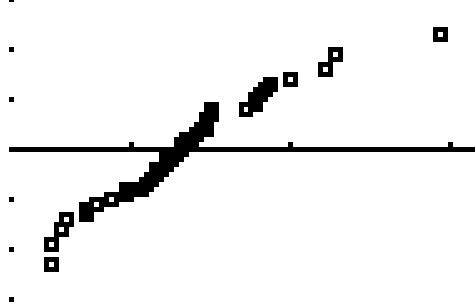
*Step 3: Verify the conditions for using the selected procedure.*

- The data comes from a sample of 46 engines from the population of all light-duty engines of this type. We are not told that the sample is an SRS, however. If the data does not come from a random sample of engines, then we should hesitate to draw conclusions about the population mean.
- Is the population distribution of NOX emissions normal? We do not know from the problem statement. A stemplot will show a roughly symmetrical, single-peaked distribution so we

can assume it is roughly normal OR the calculator normal probability plot of NOX sample data (shown below) is roughly linear, so the data is normally distributed.

```

Plot1 Plot2 Plot3
On Off
Type: L1 L2 L3
      M1 M2 M3
Data List: L1
Data Axis: Y
Mark: □ +
  
```



- Since our sample size is 46, we will assume that the populations of all engines of this type is at least 460.

*Step 4: If the conditions are met, carry out the inference procedures and interpret your results.*

a. Since we were given data, we must find our 1-variable statistics.

```

1-Var Stats
x̄=1.328695652
Σx=61.12
Σx²=91.7688
Sx=.4843993242
σx=.4791051836
↓n=46
  
```

The confidence interval formula is now

$$\bar{x} \pm t^* \frac{s}{\sqrt{n}}$$

We use the  $t$  distribution with  $df = 46 - 1 = 45$ . Unfortunately, there is no row corresponding to 45 degrees of freedom in Table C. Instead, we use the  $df = 40$  row, which will yield a higher critical value  $t^*$ , and thus a wider confidence interval. At a 95% confidence interval, the critical value is  $t^* = 2.021$ . So the 95% confidence interval for  $\mu$  is

$$\bar{x} \pm t^* \frac{s}{\sqrt{n}} = 1.329 \pm 2.021 \left( \frac{0.484}{\sqrt{46}} \right) = 1.329 \pm 0.144 = (1.185, 1.473)$$

This means that we are 95% confident that the true mean level of nitrogen oxides emitted by this type of light-duty engine is between 1.185 grams/mile and 1.473 grams/mile.

Our handy-dandy calculator can do this for us.

Select “Stat,” “Tests,” #8 “TInterval.” Since we were given the data, select “Data,” then select the confidence interval as a decimal. Calculate.

```

TInterval
Inpt: DATA Stats
List: L1
Freq: 1
C-Level: .95
Calculate

TInterval
(1.1848, 1.4725)
x̄=1.328695652
Sx=.4843993242
n=46
  
```



NOTE: If we are given the statistics instead of the data, we would simple select “Stats” instead of “Data.”

b. Here we are trying to make a claim based on the data, so we are talking about testing the hypothesis.

Since our sample mean was 1.329 grams/mile and the greatest acceptable value is 1.2 grams/mile, we are wondering if our sample indicates that our batch on engines exceeds the required limit, and should therefore be tossed out. This is a one-sided test.

$$H_0 : \mu = 1.2$$
$$H_a : \mu > 1.2$$

We need to calculate our critical  $t$  value test statistic, then find the area in the tails ( $p$ -value) based on this critical value, then compare it to the given alpha values of  $\alpha = 0.05$  and  $\alpha = 0.01$ .

$$t = \frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{\bar{x} - \mu}{SE} = \frac{1.329 - 1.2}{0.484/\sqrt{46}} = 1.807687124...$$

Looking across the row for  $df = 40$  (since we don't have  $df = 46$ ), we see that 1.8 falls between the given  $t$  values of 1.684 and 2.021. The best we can say is that the  $p$  value is between 0.05 and 0.025, but that is enough to answer the question.

This means that our result is NOT significant at the 1% level, but it IS significant at the 5% level.

Again, our number-crunching calculator can give us better results:

Select “Stat,” “Tests,” “T-test” and type in your information. Since we have our information given as data, select “data.” Leave “Freq” equal to 1. Select “Calculate”

```
T-Test
Inpt: Data Stats
μ0: 1.2
List: L1
Freq: 1
μ: ≠μ0 <μ0 >μ0
Calculate Draw
```

```
T-Test
μ>1.2
t=1.801935587
P=.03912839
x̄=1.328695652
Sx=.4843993242
n=46
```

From this screen, we get the exact  $t$ - and  $p$ -values. Notice the  $t$  test statistic is very similar to ours using  $df = 40$  instead of 46. The actual  $p$ -value is 0.039. This means that our results from our sample could have happened strictly by chance, but only about 4% of the time.

NOTE: If we are given the statistics instead of the data, we would simple select “Stats” instead of “Data.”

NOTE 2: The above calculator commands work for normal  $z$ -tests as well, just choose “Z-Test.”