4.2 Area

TOOTLIFTST:

- Understand an use sigma notation for sums
- Understand the need for Area approximation
- How to approximate areas of plane regions
- Find Upper and Lower bounds for areas of plane regions
- Find the Exact area of a Region in a plane using limits

Intro:
In the first half of calculus, we were primarily interested in finding the slope of a line at a single point. Before we learned about the derivative, or its shortcut rules, we intuitively developed a way to approximate this by taking smaller and smaller changes in y-values of increasingly smaller x-value differences. Once we took the limit of these secant slopes, we finally achieved the actual slope of the tangent line, or the instantaneous rate of change. The eventual shortcut rules made the using the formal definition a matter of preference.

The second half of calculus is the study of the integral, which ends up being an area rather than a slope. Now we geometric formulas for finding areas of nice shapes, but what if we wanted to find the area that had an irregularly shaped side, or two, or three? We will again develop the idea of area intuitively by approximation. Rather than taking small y increments **divided** by small x increments, we will be taking infinitesimally small y increments **times** small x increments. In this sense, differentiation and integration are inverse operations. Late, with the help of the limit, we will find the actual area under a curved surface. This will be long and cumbersome. Luckily, again, there are some nice shortcut rules, thanks to the Fundamental Theorem of Calculus.

Because we will be adding many (many, many, many) small area approximations together, too many too enumerate, we will develop a new symbol: the capital Greek letter for "S" of Sigma, $\sum$, which means "summation"

<u>Sigma Notation</u>
The sum of n terms $a_1 + a_2 + a_3 + \cdots + a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where i is the **index of summation**, a_i is the i **th term** of the sum, and the **upper and lower bounds of summation** are n and 1, respectively.

Note: the upper and lower sum are fixed with respect to the index, but can be anything, so the index doesn't have to be denoted with an i , nor does it have to start with the number one.

Example:
Carl Freidrich Gauss is considered one of the three greatest mathematicians of all time (with Archimedes and Newton). Legend has it that in Elementary school, Gauss's teacher was looking for a way to keep the students occupied for a very long time, so she had then find the sum of the numbers from 1 to 100. If she were to give the problem to the students in sigma notation, it would have been as follows: $\sum_{n=1}^{100} n$. The answer, of course is 5050. But what does this have to do with Gauss? Well he figured out the algebraic formula for evaluating the sum of the first n numbers, not just the first hundred! He reduced the problem to simple algebraic substitution, rather than physically adding them up. How'd he do it? He noticed that if

you write out the numbers twice, one in icreasing order, the other in decreasing order, then look at the vertical sums of each term, a pattern emerged.

$$\begin{array}{ccccccccc} 1 & + & 2 & + & 3 & + & \cdots & + & 100 \\ 100 & + & 99 & + & 98 & + & \cdots & + & 1 \\ \hline 101 & + & 101 & + & 101 & + & \cdots & + & 101 \end{array}$$

He realized he now had exactly 100 values of 101. He knew this was now a multiplication problem: $(100)(101)$. This was equal to 10100. He also realized that this sum was exactly twice the sum the teacher desired, since he had introduced an extra series of 1 to a hundred. The final answer, then was 10100/2 = 5050. Needless to say, the teacher was astonished. She new better next time to ask Gauss to paint her house with a 1 inch brush if she wanted to keep him busy.

Gauss's brilliant, precocious discovery generalizes to the following:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Finding algebraic formulas to express the sum of many terms is not always easy or even possible, but if one can be found, it is fabulous, for it eliminates the need to add the terms, which can run into the thousands (just ask Archimedes: appropriately know as the ***method of exhaustion***). The sum is now a function of the upper bound and can be evaluated by substitution. If Gauss could discover this formula, you can memorize it. You will actually have to memorize three more.

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} n^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{i}^3 = \frac{n^2(n+1)^2}{4}$$

OK, so now you've memorized these, now what? Let's see how they are useful. We are not limited to expressions of this form. Using algebra, we can simplify larger, unfamiliar expressions so that, hopefully, we can use some of the above formulas.

Ex) Evaluate $\sum_{i=1}^{n} \frac{i+3}{n^2}$ for $n = 10,\ 1{,}000,\ 100{,}000$.

Solution: (Note: you can pull scarlar / constant multiple in front of the sigma, and you can "distribute" the sigma to each term)

$$\begin{aligned} \sum_{i=1}^{n} \frac{i+3}{n^2} &= \frac{1}{n^2}\sum_{i=1}^{n}(i+3) \\ &= \frac{1}{n^2}\left(\sum i + \sum 3\right) \\ &= \frac{1}{n^2}\left(\frac{n(n+1)}{2} + 3n\right) \\ &= \frac{1}{n^2}\left(\frac{n^2+7n}{2}\right) \\ &= \frac{n+7}{2n} \end{aligned}$$

Now we can solve by substituting in the different values of n.

n	$\sum_{i=1}^{n}\frac{i+3}{n^2}=\frac{n+7}{2n}$
10	0.85
1,000	0.5035
100,000	0.500035

Notice that for $n=10$, it is not too difficult to list the ten terms in the **series** and add them together. We can't say the same for $n=100{,}000$. Are you convinced how nice the algebraic formula is? Also, notice that the sum appears to approaching 0.5 or $\frac{1}{2}$. That is $\lim_{x\to\infty}\frac{n+7}{2n}=\frac{1}{2}$. This is consistent with what we learned about limits at infinity. If you substitute x with n, you can analyze the degree of the numerator and denominator. In this case, since they are both of degree one, the limit is the leading coefficient over the leading coefficient. How does this manifest itself graphically? Remember? There will be a horizontal asymptote at $y=\frac{1}{2}$. Remember, we are talking about the limit of a sum. So, in essence, the more terms we add to the series, the sum gets closer and closer to a specific number. We say such a sum **converges**. A series or sum that increases or decreases without bound, or oscillates between two values is said to be **divergent**.

So what does all this have to do with Area and Integration? I'm glad you asked.

Let's say we have a function $f(x)=-x^2+5$. We want to find an approximation of the area of the Region bounded by the curve and the x-axis on the interval $[a,b]=[0,2]$.

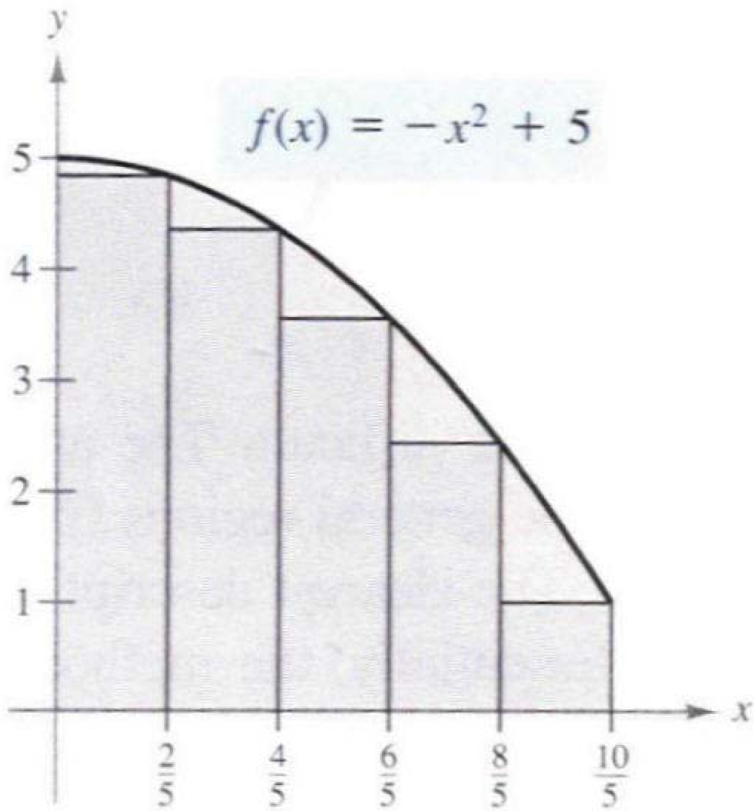
To do this, we are going to divide the region in several rectangle of equal width whose height will be determined by the function values. The more rectangles we use, the more accurate we will be. For now, let's use 5 rectangles. We are going to have the option of using either the Right End Points to determine each rectangle's height, or the Left End Points.

First, we find the width of each triangle. The **interval width** is $b-a$, 2 – 0 = 2 units (Right end pt – Left end pt). We divide this by the **number of rectangles** we desire, n, in this case 5, to achieve the uniform width of each rectangle. Call this width Δx (delta x). So, we get the equation

$$\Delta x=\frac{b-a}{n}=\frac{2-0}{5}=\frac{2}{5}$$

a) Using the Right End Points of the intervals to calculate each rectangle's height, we calculate the function value at each of these points. These x-values, c_i ,can be listed as $\frac{2}{5}i$, where $i=1,2,3,4,5$. (Realize that $\frac{2}{5}$ is the interval width AND the first x-value. The index counts the multiples of the interval width.) Remembering the area of a rectangle is width timed height, the sum of each of the 5 areas can be represented by the following:

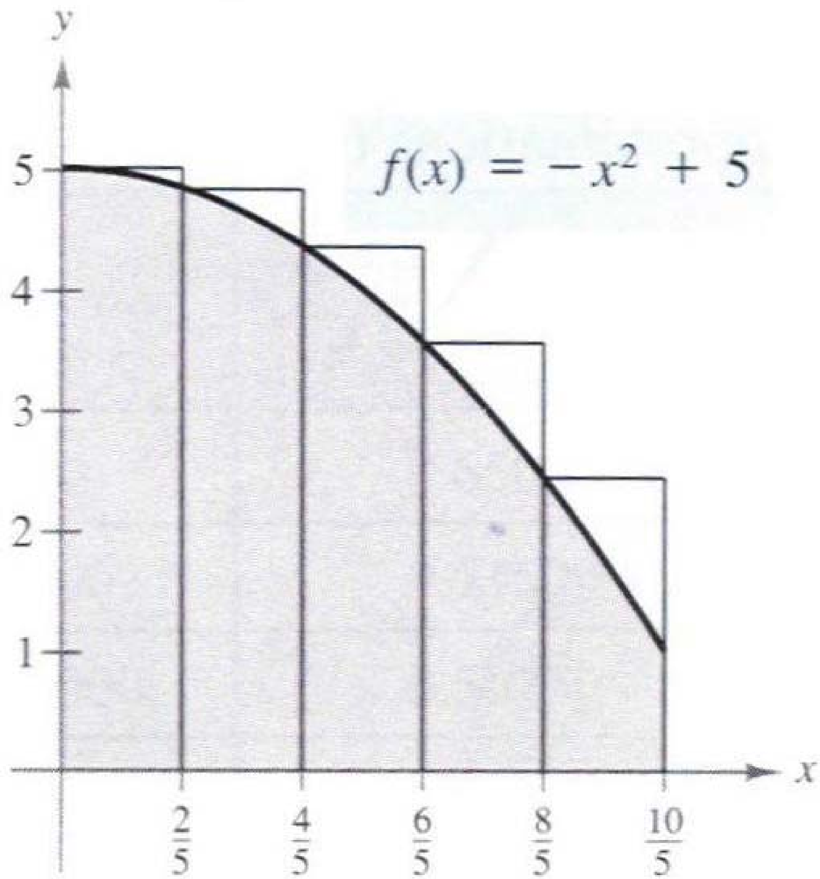
$$\sum_{i=1}^{5}(h)(w)=\sum_{i=1}^{5}f(c_i)\Delta x=\sum_{i=1}^{5}f\left(\frac{2}{5}i\right)\left(\frac{2}{5}\right)=\sum_{i=1}^{5}\left[-\left(\frac{2}{5}i\right)^2+5\right]\left(\frac{2}{5}\right)=-\frac{8}{125}\sum i^2+\sum 2=\frac{162}{25}=6.480$$



Notice the rectangles generated using the right end points yield **inscribed** rectangels giving a sum that is LESS than the actual area of the region, hence, we call this sum the **Lower Sum**. Also notice we still didn't actually need to calculate five different areas then add them together. We set up the sum, substituted the algebraic formula, then plugged in our value of $n = 5$.

b) Using the Left End Points of the intervals to calculate each rectangle's height, we calculate the function value at each of these points. These x-values can be listed as $\frac{2}{5}i - \frac{2}{5} = \frac{2i-2}{5}$, the sum of each of the 5 areas can be represented by the following:

$$\sum_{i=1}^{5} f\left(\frac{2i-2}{5}\right)\left(\frac{2}{5}\right) = \sum_{i=1}^{5}\left[-\left(\frac{2i-2}{5}\right)^2 + 5\right]\left(\frac{2}{5}\right) = \frac{202}{25} = 8.050$$



Notice the rectangles generated using the left end points yield **circumscribed** rectangles, giving a sum that is GREATOR than the actual area of the region, hence, we call this sum the **Upper Sum**.

Combining the two results from above, we conclude that

6.480 < Area of the Region < 8.080

This gives a margin of error of 8.080 – 6.480 = 1.600 square units

How could we increase our accuracy, ie, narrow our margin of error? (increase the value of n, the number of rectangles)

It is important to note that the right end points will not always achieve the Upper Sum, not will the Left end points always yield the Lower Sum. It depends on the curve and the given interval.

Now for the fun part, getting rid of the approximation by introducing the LIMIT. Yipee!

I don't think I have to convince you that we can get better and better approximations of the area by taking smaller and smaller and smaller rectangle, that is rectangle of increasingly small widths, Δx . How small can they be? You guessed it, they can approach zero**. As the rectangle widths get infinitesimally small we generate an infinite number of subintervals (or rectangles), that is** $\lim_{\Delta x \to 0}$ **implies** $\lim_{n \to \infty}$ **, and the upper and lower sums approach each other, and they both approach the actual area.** Reread that last sentence. It is an important theorem which will lend itself nicely for our formal definition of Area of a Region in the Plane.

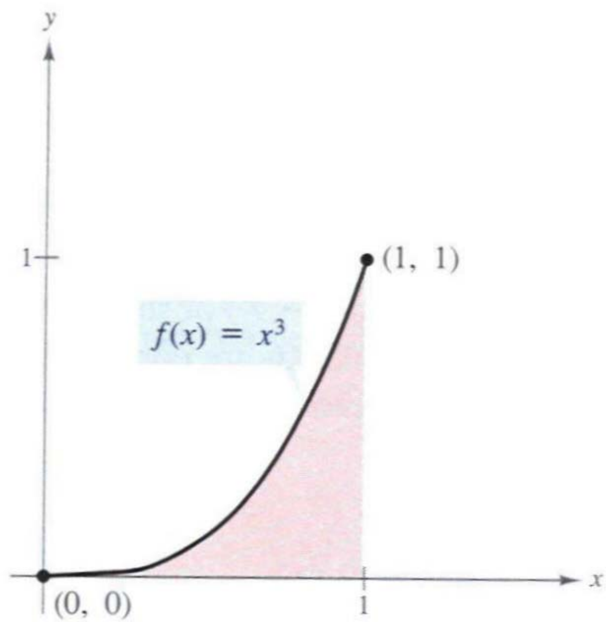
Definition of the Area of a Region in the Plane
Let f be continuous and nonnegative on the interval $[a,b]$. The area of the region bounded by the graph of f, the x-axis, and the vertical lines $x=a$ and $x=b$ is

$$\text{Area}=\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$$

Where $\Delta x=(b-a)/n$

Let's try one:

Find the area of the region bounded by the graph of $f(x)=x^3$, the x-axis, and the vertical lines $x=0, x=1$ as shown below.



Notice that the function is continuous and non-negative (we will deal with negative y-values later). Find the width of each subinterval (we don't want to really call them rectangles any more). This will be $\Delta x=1/n$.

Here is an important step: Since we will be taking the limit, and in doing so, the Upper Sums and Lower Sums converge to the actual sum, it doesn't matter if we choose the Left or the Right endpoints!! To make things easier **ALWAYS CHOOSE THE RIGHT END POINT!!!!!** This will guarantee that $c_i=a+\Delta x i$

We get our "stopping points (x-values)" to be $c_i=0+\frac{1}{n}i=i/n$

Now that we have found our Δx and our c_i, we are ready to areate (pronounced ariate—not really a word)

$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^3\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n} i^3$$

$$= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right]$$

$$= \lim_{n\to\infty}\left(\frac{1}{4}+\frac{1}{2n}+\frac{1}{4n^2}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1}{4}+0+0\right)$$

$$= \frac{1}{4}$$

From this we can conclude that the area of the Region is $\frac{1}{4}$ . . . Exactly. No more, no less.

Ta Da!

All images and some examples are taken from the Calculus text by Larson, Hostetler, and Edwards, fifth edition.