4.3 Riemann Sums and Definite Integrals

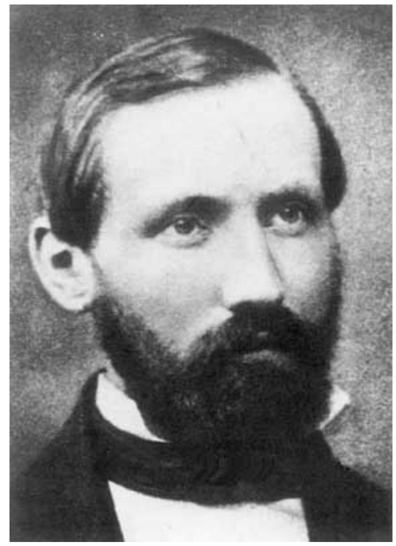
TOOTLIFTST:

- Compute Riemann sums using left, right, and midpoint evaluation points.
- Find definite integrals as a limit of Riemann sums over equal subdivisions
- Use Riemann sums to approximate definite integrals of functions represented algebraically, geometrically, and by a table of values.

Introduction

What we did in the last section was found areas using subintervals of equal width . . . but they don't have to be! A man named Georg Friedrich Bernhard Riemann, a German mathematician, said, "Vat if vee let di vidth of each subinterval get smaller und smaller und smaller as n approached invinity?" This was a logical question. The result is that you still get an infinite number of subintervals, each of infinitesimally small width x_i, but . . . you get more than if they were all the same width AND it takes longer to fill up the interval. Sounds weird, I know, but listen to this: **As n increases, the length of the largest subinterval approaches zero.** This subtle distinction in allowing the interval width to be non-specific is very important of what we are going to call the **Definite Integral**.

Whenever you find areas using the limit of the sum of the areas of rectangles, regardless of their width, and whether we end up finding them algebraically or numerically, we call this method: **<u>Riemann Sums</u>**



Georg Friedrich Bernhard Riemann
Courtesy of MacTutor Archives

Let's develop the Definite Integral for finding the Area of a Region in a Plane.

We call the largest subinterval the Norm, denoted as $\|\Delta\| = \Delta x = \dfrac{b-a}{n}$

If we let the norm approach zero, it is the same as letting n approach infinity. Think about that and convince yourself.. We will use this when taking the limit of the sum to obtain the definition of the Definite Integral.

Definition of a Definite Integral

If f is defined on the closed interval $[a,b]$ and the limit

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

exists, then f is **integrable** on $[a,b]$ and the limit is denoted by

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx\ .$$

The limit is called the definite integral of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

This basically means that the Definite Integral is a Definite number, not an indefinite expression or family of functions (+C), that is equal to the **area** under the curve from *a* to *b*.

So, what types of curves can we find the area underneath? Well, it sure is hard to find the area under a removable discontinuity, isn't it? It's hard to be under something that isn't there. This leads us to a sufficient condition for integrability.

Theorem: Continuity Implies Integrability
If a function f is continuous on the closed interval $[a,b]$, then it is integrable on $[a,b]$.
So, C $\Rightarrow$ I

Notice the converse is not true. A function can be integrable without being continuous, for example $f(x)=[\![x]\!]$ on $[0,2]$. The difference here is that, although there is a discontinuity, it is non-removable, in other words, the y-values still exist there.
It is also worth restating here that Differentiability implies Continuity, or D $\Rightarrow$ C. Joining the two statements by the transitive property, we can conclude that Differentiability implies Integrability or D $\Rightarrow$ I. Is it true that I $\Rightarrow$ D?

Let's work an example:

Evaluate the definite integral $\int_{-2}^{1} 2x\,dx$.

Notice that what this is really asking us to do is find the area under the curve, with respect to the x-axis between –2 and 1. The function is a line, and thus continuous over our interval $[-2,1]$. Using the definition of the definite integral, we need to find our c_i and our Δx. We end up doing the exact same thing we did in the last section.

$\Delta x=\frac{1-(-2)}{n}=\frac{3}{n}$. $c_i=-2+\frac{3i}{n}$. Plugging into the definition, we get the following:

$$\int_{-2}^{1} 2x\,dx = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(c_i)\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n} 2\left(-2+\frac{3i}{n}\right)$$

$$= \lim_{n\leftarrow\infty}\frac{6}{n}\sum_{i=1}^{n}\left(-2+\frac{3i}{n}\right)$$

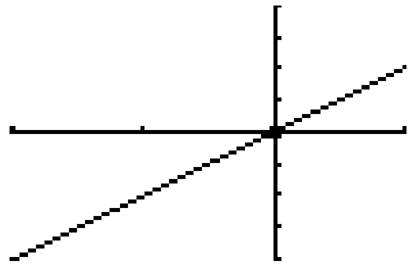
$$= \lim_{n\to\infty}\frac{6}{n}\left\{-2n+\frac{3}{n}\left[\frac{n(n+1)}{2}\right]\right\}$$

$$= \lim_{n\to\infty}\left(-\frac{12n}{n}+\frac{18n^2+18n}{2n^2}\right)$$

$$=(-12+9)$$

$$=-3$$

What does it mean to have negative area? Remember, we defined the definite integral as the area with respect to the x-axis. We designate the area **above the x-axis as positive** and **below it as negative**. If the area comes out negative, more of the area lies below the x-axis than above it. In this sense, the definite integral gives us the ***NET*** area under the curve. Let's verify this algebraically. The graph looks like



Since it is a straight line, we can think of it as two separate triangles, one below and one above the x-axis. The Area of the lower part is $A = 1/2bh = -(1/2)(2)(4) = -4$, remember it's below the x-axis so it's negative. The Area of the upper part is $A = (1/2)(1)(2) = 1$. The NET area is their sum, which is –3.

How could you find the ***Gross Total*** area? This is the squared footage number we would need if we were trying to paint the region. This would be, from above, 4 + 1 = 5. How would we find it using calculus? This would require you to find points of intersection and integrate on the intervals between them. The following properties will help you do that.

1. If f is defined at $x = a$, then $\int_a^a f(x)dx = 0$
2. if f is integrable on $[a,b]$, then $\int_b^a f(x)dx = -\int_a^b f(x)$
3. if f is integrable on the three closed intervals determined by a,b,c then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Notes:

1. This means the area under a single point is zero. This makes sense because it is one-dimensional. It has no width and, therefore, no area.
2. This implies that integration is directional. It is traditionally done from left to right. If we integrate in the opposite direction, we obtain an area of equivalent magnitude, but opposite in sign.
3. This means that I can find the area on an interval by taking the sum or difference of two sub or overlapping intervals respectively. Notice that c need to be between a and b.

Let's use this to find the Net area from the previous problem. The graph of $f(x) = 2x$
Has an x-intercept a $x = 0$. Combining properties 2 and 3 from above, we can write the Area as the following:

$$A_{net} = \int_{-2}^{0} 2xdx + \int_{0}^{1} 2xdx$$

$$= -\int_{0}^{-2} 2xdx + \int_{0}^{1} 2xdx$$

evaluating using infinite sums yield $A = 5$

We will do this again when we learn how to evaluate Definite integrals with the Fundamental Theorem of Calculus.