

4.4 The Mean Value Theorem for Integrals

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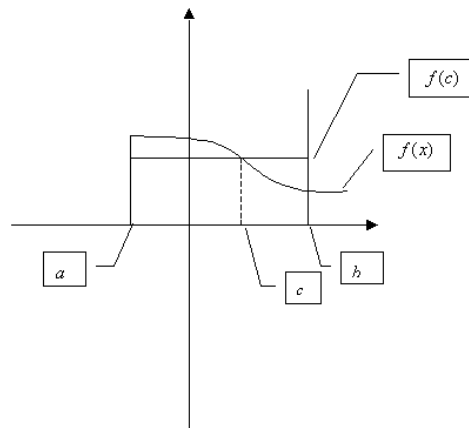
- Basic properties of Definite Integrals
- Understand the average value over an interval.

Not to be confused the Mean Value Theorem (for derivatives), the MVT for Integrals also deals with average values, not ill-tempered ones. Here's what it states:

If f is continuous on the closed interval $[a, b]$, then there exists a number c in the closed interval $[a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a)$$

Notice that the left side of the equation represents the area under the curve with respect to the x-axis on the interval from a to b . The right side of the equation is the area of a rectangle of height $f(c)$ and width $(b-a)$. What the theorem basically states is that the area under the curve is the same as the area of a rectangle over the same interval for some specific height, $f(c)$. The x-value of this height, c , is guaranteed to exist under the conditions of the hypothesis. How do we find this value of c ?



To find c , we need to find $f(c)$, and to do that, we need to solve the above equation by dividing each side by $(b-a)$ or multiplying by $1/(b-a)$.

$$\frac{1}{(b-a)} \int_a^b f(x)dx = f(c)$$

The value of $f(c)$ has a very special meaning. It is called the **average value** of f on the interval $[a, b]$. This means, if you were to add up every single y-value along the curve from a to b (which is impossible to do, since there are an infinite number of them) and divide the number of y-values you added up (infinity), this would be the number. This is the same way your six-weeks average is calculated, although we are only adding up a discrete number of grades.

To get around the little problem of adding up an infinite number of values and dividing by infinity, the MVT employs a little trick. If we could find the area of the interval (which is height times width) then divide by the width, you would end up with height, and it would be the average height, or y-value.

So, the area of the region under the graph of f is equal to the area of the rectangle whose height is the average value. Do you have it? If not, return to the top and reread. Repeat as necessary.

Example:

Find the average value of $f(x) = 3x^2 - 2x$ on the interval $[1, 4]$.

Solution: The average value is given by

$$\begin{aligned}f(c) &= \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{4-3} \int_1^4 (3x^2 - 2x) dx \\ &= \frac{1}{3} [x^3 - x^2]_1^4 \\ &= \frac{1}{3} [(64 - 16) - (1 - 1)] = \frac{48}{3} = 16\end{aligned}$$

So what is the value of c guaranteed by the theorem? All we have to do is solve the original function for x after we set it equal to 16.

$$f(x) = 3x^2 - 2x = 16$$

$$3x^2 - 2x - 16 = 0$$

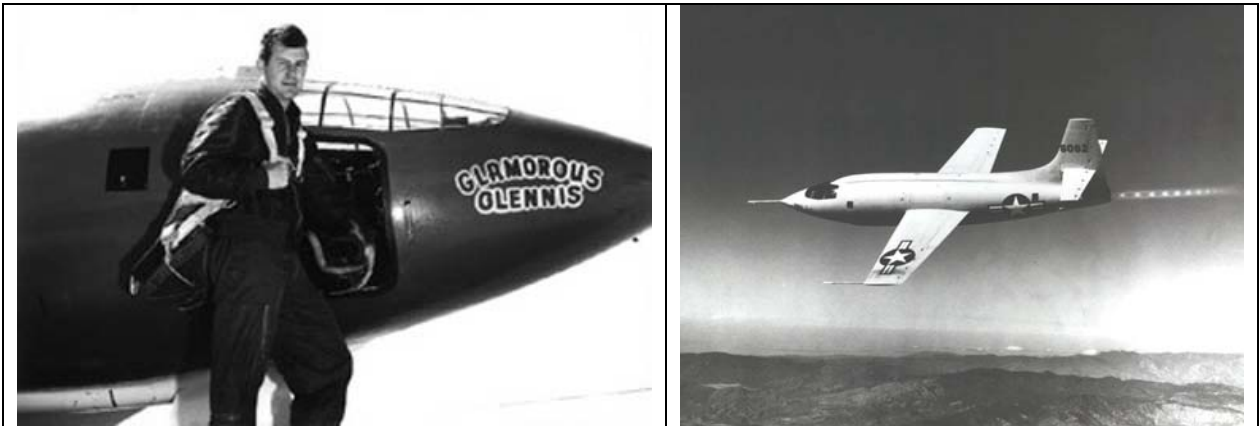
$$(3x - 8)(x + 2) = 0$$

$$x = \frac{8}{3}, -2$$

Of these two values, only $\frac{8}{3}$ or 2.667, is in our interval, hence is the desired value of c .

Here is a nice application:

The first person to fly at a speed greater than sound was Charles Yeager. ON October 14, 1947, flying in an X-1 rocket plane at an altitude of 12.8 kilometers, Yeager was clocked at 299.5 meters per second (about 669.962 mph.) If Yeager had been flying at an altitude under 10.375 kilometers, his speed of 299.5 meters per second would not have “broken the sound barrier.”



Chuck Yeager and the X-1 research plane that broke the sound barrier. It can be seen today at the Smithsonian Air & Space Museum in Washington, DC.

Courtesy Air Force Test Center History Office

At different altitudes in earth's atmosphere, sound travels at different speeds (to see why, click [here](#)). The speed of sound $s(x)$ (in meters per second) can be modeled by

$$s(x) = \begin{cases} -4x + 341, & 0 \leq x < 11.5 \\ 295, & 11.5 \leq x < 22 \\ \frac{3}{4}x + 278.5, & 22 \leq x < 32 \\ \frac{3}{2}x + 254.5, & 32 \leq x < 50 \\ -\frac{3}{2}x + 404.5, & 50 \leq x < 80 \end{cases}$$

where x is the altitude in kilometers. What is the average speed of sound over the interval $[0,80]$?

Notice the question doesn't mention anything about the integral. You must discern for yourself that since it is talking about the Average Speed, that this is finding the $f(c)$ from the MVT for integrals.

Begin by Integrating over the 5 subintervals:

$$\int_0^{11.5} s(x)dx = \int_0^{11.5} (-4x + 341)dx = 3657$$

$$\int_{11.5}^{22} s(x)dx = \int_{11.5}^{22} 295dx = 3097.5$$

$$\int_{22}^{32} s(x)dx = \int_{22}^{32} \left(\frac{3}{4}x + 278.5\right)dx = 2987.5$$

$$\int_{32}^{50} s(x)dx = \int_{32}^{50} \left(\frac{3}{2}x + 254.5\right)dx = 5688$$

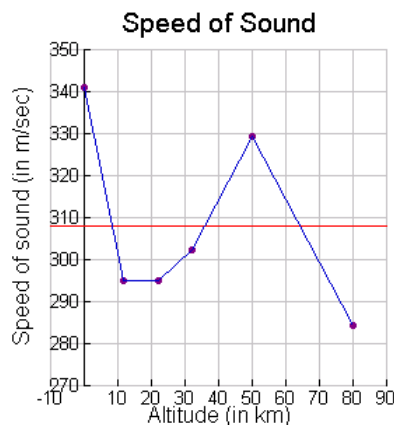
$$\int_{50}^{80} s(x)dx = \int_{50}^{80} \left(-\frac{3}{2}x + 404.5\right)dx = 9210$$

By adding the values of the five integrals, you have

$$\int_0^{80} s(x)dx = 24,640.$$

Therefore, the average speed of sound from an altitude of 0 kilometers to 80 kilometers is

$$\text{Average speed} = \frac{1}{80} \int_0^{80} s(x)dx = \frac{24,640}{80} = 308 \text{ meters per second (or about 688.976mph)}$$



We've already had so much fun, now we will double it with the Second FUNDamental Theorem of Calculus. It again, has to do with the fact that integration and differentiation are inverse operations of each other. They basically undo the work of the other. What do you think would happen if you took the integral of a function, then turned around and immediately took the derivative? (pause for appropriate time to think) That's right! You would end up with the original function. That's the intuitive idea behind the 2nd FTOC.

The Second Fundamental Theorem of Calculus

If f is continuous on an open interval I containing a , then for every x in the interval,

$$\frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x)$$

Let's dissect it in laymen's terms. The function is continuous, meaning smooth, no jumps, gaps, etc. . This also means it is integrable. The x -value a is just some value in the interval, a number. The upper bound is any other value in the interval, hence it is a variable x . To avoid using two x 's, the function in the integral is defined with t , a temporary variable.

Notice if you worked the left side from the inside out, taking the integral first and evaluating using the FTOC, you would plug in an x into the antiderivative (which gives you a function of x) then subtract the value of the antiderivative at $x = a$ (which is a constant). Then, when you take the derivative, you get the function $f(x)$ minus the derivative of a constant, which is zero. Therefore, the final result is just $f(x)$.

Let's look at an example.

Evaluate $\frac{d}{dx} \left[\int_0^x \sqrt{t^2 + 1} dt \right]$.

First we must check the hypothesis of the theorem. The function IS continuous on the open interval containing $x = 0$ (it is actually continuous everywhere.) So the theorem applies.

The answer is then, simply, $\sqrt{x^2 + 1}$! "That's so easy," you say. "What's the catch?"

The catch is, the theorem becomes especially useful when either the function is too difficult to integrate by had AND/OR the upper bound is not just x , but a function of x , in which case, the CHAIN RULE applies.

Example #2

Find the derivative of $F(x) = \int_{\pi/2}^{x^3} \cos t dt$.

Cosine is continuous for all x , and $\pi/2$ is one of these values, so the theorem applies, although we COULD actually do it as a sequence of integration and differentiation.

Notice we are still integrating from a number to *something* with an x , it just happens to be a function of x .

All we have to do is plug into t the function of x , then multiply by the derivative of the function of x .

$$\frac{d}{dx} \int_{\pi/2}^{x^3} \cos t dt = [\cos(x^3)][3x^2] = 3x^2 \cos x^3$$

Again, notice that the constant that is the lower bound of integration, as long as it is in the continuous interval of the function, has no effect on the answer. Do you remember why?

Here's a variation with a clever trick. It's fun being clever.

Example #3

Find $F'(x)$ if $F(x) = \int_{-x^2}^x t^3 dt$

There are no constants in the intervals of integration, but the function is a cubic polynomial, so we know it is continuous for all x . We need to introduce a constant in order to use the 2nd FTC. Here's how.

Recall the property: For a, b, c in a continuous interval of $f(x)$, $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$. The value of c , does not need to be between a and b .

Also, recall the additional property that $\int_a^b f(x) dx = -\int_b^a f(x) dx$

I am going to choose an arbitrary value of c from above. My choice, if it is in the continuous interval of f , is always $x = 0$.

Rewriting the original problem from the first property, we get

$$F(x) = \int_{-x^2}^0 t^3 dt + \int_0^x t^3 dt$$

Using the second property to get the constants as the lower bounds, we get

$$F(x) = \int_0^{-x^2} t^3 dt + \int_0^x t^3 dt$$

Now, we can take the derivative

$$F'(x) = \underbrace{(-x^2)^3}_{\text{plug in}} \underbrace{(-2x)}_{\text{chain rule}} + \underbrace{(x)^3}_{\text{plug in}} \underbrace{(1)}_{\text{chain rule}} = 2x^7 + x^3 \quad \text{TA DA!}$$