4.4 The Fundamental Theorem of Calculus

a.k.a the "shortcut" rule

TOOTLIFTST:

- Determine definite integrals as the rate of change of a quantity over an interval interpreted as the change of the quantity over the interval.
- Understand basic properties of definite integrals.
- Use the Fundamental Theroem of Calculus to evaluate definite integrals.
- Use the Fundamental Theorem to represent a particular antiderivative, and the analytical and graphical analysis of functions so defined.

The Fundamental Theorem of calculus relates the derivative to the integral. It basically states that differentiation and integration are inverse operations. The derivative involves a quotient (dy/dx), the integral involves a product ($dy * dx$) . The theorem states that the inverse relation ship between division and multiplication is preserved under the limit process. What it actually gives us is a convenient way to evaluate area over given intervals once we know the antiderivative.

Theorem: The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a,b]$ and F is an antiderivative of f on the same interval, then

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a) .$$

So, once we have the antiderivative, all we need to do is evaluate it at the endpoints of the interval and find the difference, Right end point minus Left end point.

Example 1:

$$\int_1^2 (x^2 - 3)dx = \left[\frac{x^3}{3} - 3x\right]_1^2 = \left(\frac{8}{3} - 6\right) - \left(\frac{1}{3} - 3\right) = -\frac{2}{3}$$

Example 2:

Find the area of the region bounded by the graph of $y = 2x^2 - 3x + 2$, the x-axis, and the vertical lines $x = 0$ and $x = 2$.

If the problem asks for the area, it means the Gross area. You always want to draw the picture first and identify the region. For this particular problem, the entire area is above the x-axis.

$$\begin{aligned}\text{Area} &= \int_0^2 (2x^2 - 3x + 2)dx \\ &= \left[\frac{2}{3}x^3 - \frac{3}{2}x^2 + 2x\right]_0^2 \\ &= \left(\frac{16}{3} - 6 + 4\right) - (0 - 0 + 0) \\ &= \frac{10}{3}\end{aligned}$$

Sometimes, we need to rewrite the original function in order to find the area.

Example 3:

Evaluate $\int_0^2 |2x-1|dx$ we recognize this to be a transformation of the "V" graph, with new vertex at $x = 1/2$. Can we integrate directly from 0 to 2? Why or why not? If we integrated then plugged in, disregarding the absolute value sign, we would be finding the NET area of the line $f(x) = 2x-1$, which has some positive and negative values. Because our interval of integration contaings the vertex, we need to set it up as two intervals to assure ourselves that all the area we find is positive.

$$|2x-1| = \begin{cases} -(2x-1), & x < \dfrac{1}{2} \\ 2x-1, & x \geq \dfrac{1}{2} \end{cases}$$

Now we can write two separate integrals and combine the results.

$$\begin{aligned}
\int_0^2 |2x-1|dx &= \int_0^{1/2} -(2x-1)dx + \int_{1/2}^2 (2x-1)dx \\
&= \left[-x^2+x\right]_0^{1/2} + \left[x^2-x\right]_{1/2}^2 \\
&= \left(-\frac{1}{4}+\frac{1}{2}\right) - (0=0) + (4-2) - \left(\frac{1}{4}-\frac{1}{2}\right) \\
&= \frac{5}{2}
\end{aligned}$$

To summarize, the Definite integral is equivalent to finding the area under the curve on a given interval. It's practical meaning is the **accumulated values over a given interval**.

Example:

For a person at rest, the rate of air intake v, in liters per second, during a respiratory cyle is

$$v = 0.85\sin\frac{\pi t}{3}$$

where t is time in seconds. Find the volume in liters of air inhaled during one cycle by integrating this function over the interval $[0,3]$.

Notice the units of the y-axis are $\dfrac{liters}{\sec}$ and the units of the x-axis are seconds. When we integrate, we multiply these two together, giving us a net unit of $\dfrac{liters}{\sec}(\sec) = liters$. Finding the definite integral will give use the accumulated air in the lungs, in liters, over the 3 second interval.

$$\int_0^3 0.85\sin\frac{\pi t}{3} = \left[-.85\left(\frac{3}{\pi}\right)\cos\frac{\pi t}{3}\right]_0^3 = \left(.85\left(\frac{3}{\pi}\right)\right) - \left(-.85\left(\frac{3}{\pi}\right)\right) = \left(\frac{2.55}{\pi}\right) + \left(\frac{2.55}{\pi}\right) = \frac{5.1}{\pi} \approx 1.623 \text{ liters}$$