4.5 Integration by Substitution

TOOTLIFTST:

We are now going to learn another way to find antiderivatives, other than the power rule. It is going to involve recognizing patterns, and "inside" and "outside" functions, similar to the chain rule.

Recall the chain rule:

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x)$$

if we take the integral of both sides, we get

$$F(g(x)) + C = \int F'(g(x))g'(x)dx$$

We state this formally as the formal theorem for Integration of a composite function.

Antidifferentiation of a composite function Let g be a function whose range is an interval, I, and let f be a function that is continuous on I. If g is differentiable on ists domain and F is an antiderivative of f on I, then $\int f(g(x))g'(x)dx = F(g(x)) + C$

In other words, if we have an integral of either a product or a quotient, and we recognize one of them being the derivative of the other, we can call the outer function, *f(u)* inner function u and the derivative of the inner function du. Rewriting from above, we get

Let $u = g(x)$, then $du = g'(x)dx$ and

$$\int f(u)du = F(u) + C$$

Changing the variables involving the "dummy" variable u is a very formal process, and is not necessary. If you can identify the "outer" function, the inner function, and the derivative of the "inner", the antiderivative will involve only the "outer" function.

We will work an example both ways. You can be the judge of which way is easier.

Pattern Recognition	u-substitution
Evaluate $\int (x^2+1)^2(2x)dx$ Notice, the "inner" function (we'll call it "something" again) is x^2+1 and its derivative, $2x$, is on the outside. "Something" squared would be the "outer" function. Our *guess* for the antiderivative involves the "something" squared using the power rule. Guess: $\frac{(something)^3}{3}+C=\frac{(x^2+1)^3}{3}+C$ Derivative Check: $\underbrace{\frac{3(x^2+1)^2}{3}}_{\text{power rule}}\underbrace{(2x)}_{\text{chain rule}}$ Our guess is correct. Sometimes, we may need to adjust our guess to account for a scalar multiple or value of x. So the answer is $\frac{(x^2+1)^3}{3}+C$	Evaluate $\int (x^2+1)^2(2x)dx$ Let $\begin{array}{l} u=x^2+1 \\ du=2xdx \end{array}$ Replacing in the original expression $\int u^2du=\frac{u^3}{3}+C$ Replacing u back into the formula, we get $\frac{(x^2+1)^3}{3}+C$

Although the right side looks more condense, it is a formal procedure that requires you to write down each step. The left side, with practice, can become much quicker. You will be able to do it by sight, requiring you only to write down the answer.

Example #2 by pattern

Evaluate $\int 5\cos 5x dx$

Here the "inside" function is $5x$. The derivative, 5, is on the outside. Our guess then becomes the integral of the outside function, cos (something).

Guess: $\sin(5x)$

Deriv check: $5\cos(5x)$ TA DA!

OK, so what if there is a correction to make? I'm glad you inquired.

Example #3

Evaluate $\int x(x^2+1)^2 dx$

The derivative of the inside function, x^2+1, is $2x$, which is "kind of" on the outside, if it weren't for the unwanted scalar multiple of 2. What if our guess had another scalar multiple there, waiting to intercept or "correct" the unwanted 2? Let's try.

Guess: $\underbrace{\left(\frac{1}{2}\right)}_{\text{correction}} \frac{(x^2+1)^3}{3}+C$

Deriv check: $\frac{3(x^2+1)^2}{(2)(3)} \underbrace{(2x)}_{\text{chain rule}} = x(x^2+1)^2$ TA DA!

The moral of the story is this: "If thou is off by only a scalar multiple, thou can correct your guess by multiplying by the reciprocal of the aforementioned unwanted scalar multiple."

Try out the following example (#4)

Evaluate $\int \sqrt{2x-1}dx$

Rewriting first, we get $\int (2x-1)^{1/2} dx$

The derivative of the inside function $2x$-1 is 2, an unwanted scalar multiple. We can get so confident in our guesses, that we don't even need to check them. Here's our guess, the CORRECT guess.

The answer is $\underbrace{\frac{1}{2}}_{\text{correction}} (2x-1)^{3/2} \underbrace{\left(\frac{2}{3}\right)}_{\text{power rule}} + C = \underbrace{\frac{1}{3}\sqrt{(2x-1)^3}}_{\text{rewriting}} + C$

Are you getting the hang of it? Good! That means it's time to mix it up a bit.
You are naturally asking yourself, "What if we are off by more than just a scalar multiple, say, an x," The answer lies in the next example.

Example #5

Evaluate $\int x\sqrt{2x-1}dx$

This example is similar to above, but now we get the unwanted 2, AND we have the unwanted x. CRIMONY, DRATS, FIDDLESTICKS. No Fear. There is a method. Here's a rule of thumb:

When you are off by more than just a scalar multiple AND the "inside" function is linear, try using u-substitution

Let $u=2x-1$ and $du=2dx$. So $dx=\frac{1}{2}du$

When we plug in, we'll need to get rid of the x in terms of u. All we have to do is solve our above equation for x.

$$x=\frac{u+1}{2}$$

Now plug in all your new variables.

$$\int\left(\frac{u+1}{2}\right)\left(\sqrt{u}\right)\left(\frac{1}{2}du\right)=\int\frac{1}{4}(u+1)\left(u^{1/2}\right)du=\frac{1}{4}\int\left(u^{3/2}+u^{1/2}\right)du=\frac{1}{4}\left[\left(\frac{2}{5}\right)\left(u^{5/2}\right)+\left(\frac{2}{3}\right)\left(u^{3/2}\right)\right]+C$$
$$=\frac{1}{10}\sqrt{u^5}+\frac{1}{6}\sqrt{u^3}+C$$

Plugging in our original value for u, we get

$\frac{1}{10}\sqrt{(2x-1)^5}+\frac{1}{6}\sqrt{(2x-1)^3}+C$ TA DA! Nothing to it.

What if the integrand involves a trig function that is not a known derivative? If it's derivative is on the "outside," then we can use the power rule, too.

Example #6

Evaluate $\int \sin^2 3x \cos 3x dx$

I strongly recommend you get in the habit of rewriting so that you can see the "inside" function. Doing this, we get

$$\int (\sin 3x)^2 \cos 3x dx$$

Notice the derivative of the inside function is on the outside, off by only an unwanted "3." Our guess involves the "outer" function. We get

$$\underbrace{\left(\frac{1}{3}\right)}_{correction}\ \underbrace{\frac{(\sin 3x)^3}{3}}_{power\ rule}+C$$

Notice how this is different from $\int (\sin 3x) dx$ where the derivative of the "inside" function is not on the "outside". This integral becomes $-\frac{1}{3}\cos 3x + C$.

When evaluating definite integrals, you find the antiderivative the same way as above, you just don't have to write the "+C." Do you remember why? You can then evaluate it on the defined interval using the Fundamental Theorem of Calculus. But what if you are forced to change variables, when you are off by more an x.

Example #7

Evaluate $A=\int_1^5 \frac{x}{\sqrt{2x-1}}dx$

Solution:

Let $u=2x-1$. Therefore, $du=2dx$ so $dx=\frac{1}{2}du$ and $x=\frac{u+1}{2}$

Substituting these values in, we get

$$A = \int \frac{\frac{u+1}{2}}{\sqrt{u}} \left(\frac{1}{2} du \right)$$ What about the intervals of integration? We have two options. We can continue on as if it were an indefinite integral and substitute our value of u back in at the end, then evaluate on the original x-intervals, OR we can leave it in terms of u, which means we need to plug the original x-intervals of integration into our equation that relates u to x. Doing this, we get

$$u = 2(1) - 1 = 1$$
$$u = 2(5) - 1 = 9$$

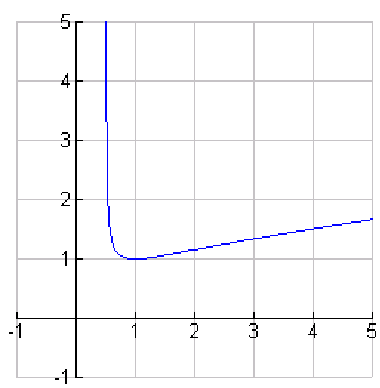
Cleaning up the integral above and putting in our new intervals of integration in terms of u, we get

$$A = \frac{1}{4}\int_1^9 (u+1)\left(u^{-1/2}\right)du = \frac{1}{4}\int_1^9 \left(u^{1/2} + u^{-1/2}\right)du$$
$$= \frac{1}{4}\left[\frac{2}{3}u^{3/2} + 2u^{1/2}\right]_1^9 = \frac{1}{4}\left[\left(\left(\frac{2}{3}\right)(27) + (2)(3)\right) - \left(\left(\frac{2}{3}\right)(1) + (2)(1)\right)\right]$$
$$= \frac{1}{4}\left[(18+6) - \left(\frac{2}{3} + 2\right)\right] = \frac{1}{4}\left[24 - \frac{8}{3}\right] = 6 - \frac{2}{3} = \frac{16}{3} \approx 5.333$$

This problem, done back in terms of x would have yielded the same answer, although it would have been a little longer.

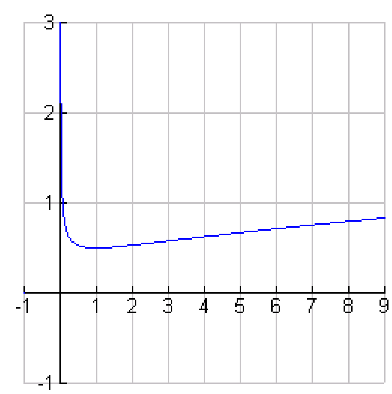
What we have essentially done above is define two DIFFERENT functions above, one in terms of x and one in terms of u, over DIFFERENT intervals that had the exact same area. This can be illustrated by the two graphs below.

In terms of x



The area under this curve from 1 to 5 is $\frac{16}{3}$

In terms of u



The area under this curve from 1 to 9 is also $\frac{16}{3}$.

Finally, (we are almost there) we will finish up with two important properties which we have already informally discussed in class.

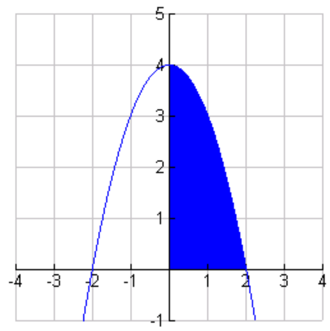
Integration of Even and Odd Functions

Let f be integrable on the closed interval $[-a, a]$.

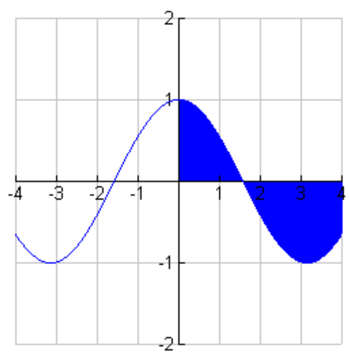
1. If f is and EVEN function, then $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.

2. If f is and ODD function, then $\int_{-a}^{a} f(x)dx = 0$.

For the first one, since Even functions have y-axis symmetry, and we are integrating on a symmetrical interval, the areas will be equivalent on either side of the y-axis, so we can just double half of it. Why would we want to do this? It is often much easier to evaluate functions, in this case antiderivatives at $x = 0$ rather than at a different number. Letting the lower bound of integration be zero, we simplify the computations.



For the second one, since Odd functions have origin symmetry, and we are again integrating on a symmetrical interval, the areas will again be equal, but opposite in sign. This means they will always cancel each other out.



Always be aware of the characteristics of the functions you are working with. Making key connections and insights can dramatically simplify certain procedures!!

The End