4.6 Numerical Integration: The Trapezoidal Rule

TOOTLIFTST
- Numerical approximations to definite integrals
- Use of trapezoidal sums to approximate definite integrals of functions represented algebraically, geometrically, and by tables of values.

Numeric Integration is useful for finding area approximations. Why would we want an approximation, though, rather than the actual area (the limit of Riemann sums and Definite Integrals can do this for us if we have the equation)? Sometimes, all we need is an approximation. Sometimes we have data that cannot be represented by a particular function. Sometimes we just have a graph, and we can quickly use geometric shapes to find areas quickly. Even if we did have the equation of the desired function, sometimes it can either be too difficult to integrate by hand, or not integrable at all! This is when numeric integration is useful (without a calculator of course.)

So far we have learned a couple of ways to approximate areas numerically. Here's a summary:

1. By partitions that can be calculated using Geometric formulas. Ex) Circles, Triangles, Rectangles, Squares, and Trapezoids.
2. By Riemann Sums with *n* equal subintervals using either Left-endpoints, Right-endpoints, or Midpoints.

We will now learn another method that, for a discreet number of subintervals, yield a better approximation of area than Riemann sums. It uses the TRAPEZOID.

Let's review the definition of a trapezoid from your Geometry class.

A trapezoid is a quadrilateral (has 4 sides) and has only one pair of sides parallel.

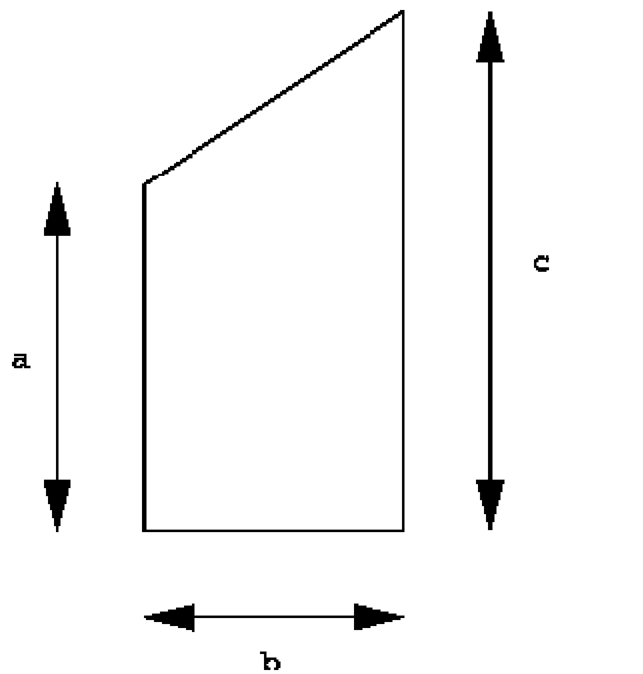
How to determine the area of a trapezoid:

- **Add the lengths of the 2 parallel sides**
- **divide by 2 to get the average length of the parallel sides.**
- **Multiply this by the height (distance between the parallel sides)**

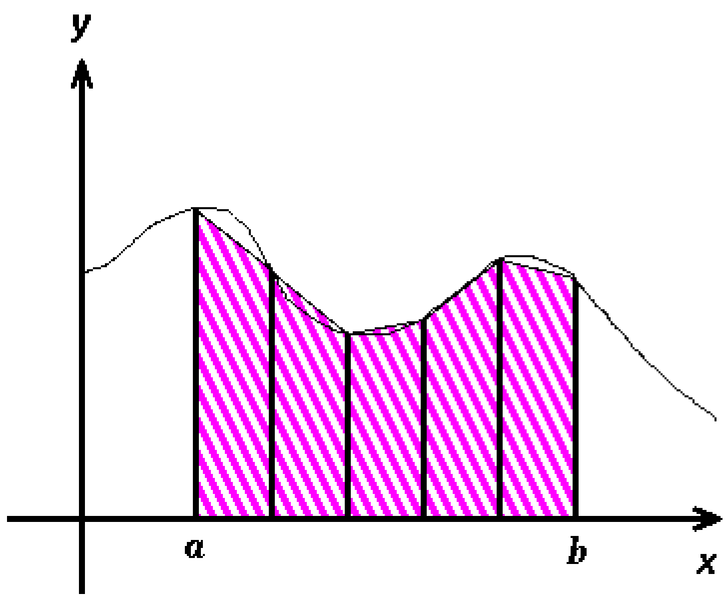
Notice this trapezoid is tilted up on its side. The “bases” are the two parallel sides, *a* and *c*. The “height” is *b*. The area of the trapezoid becomes

$$A = \frac{1}{2}(a + c)b \, .$$

When we use trapezoids on intervals under a function curve, we get a better approximation than if we used rectangles, however, we need to use the function values at both the left and right endpoint, rather than just one of them. The interval width becomes the “height” in the formula.



The following diagram will illustrate how the interval from *a* to *b* is subdivided into subintervals of equal length using trapezoids. The width of each subinterval, Δx , is then found by the following formula: $\Delta x = \dfrac{b - a}{n}$.



To find the area approximation, we find the area of each trapezoid and add them up. The stopping points of the intervals will be similar to those from Riemann Sums:

$$c_i = a + \Delta x i \text{ , where } c_0 = a$$

The formula then becomes:

$$A=\sum_{i=0}^{n}\frac{1}{2}\left(f(c_i)+f(c_{i+1})\right)\Delta x$$, where $\frac{1}{2}$ and Δx are constants

For the above graph, specifically, we can expand the sum to look like

$$A=\frac{1}{2}[f(c_0)+f(c_1)]\Delta x+\frac{1}{2}[f(c_1)+f(c_2)]\Delta x+\frac{1}{2}[f(c2)+f(c_3)]\Delta x+\frac{1}{2}[f(c_3)+f(c_4)]\Delta x+\frac{1}{2}[f(c_4)+f(c_5)]\Delta x$$

Notice that each term has two common factors, $\frac{1}{2}$ and Δx, AND all the function values except the endpoints appear TWICE. This is because they each act as the right endpoint of the left trapezoid and the left endpoint of the next one. Recognizing this, we can simplify the above formula to the following:

$$A=\frac{1}{2}\Delta x[f(c_0)+2f(c_1)+2f(c_2)+2f(c_3)+2f(c_4)+f(c_5)]$$

If we remember the ½ out in front every time and to double the middle function values, the formula is very similar to the Riemann sum approximations.

How can we increase the accuracy of our approximations? That's right! We take more and more trapezoidal subintervals with smaller and smaller "heights," our Δx. To see this happen very quickly, click here.

Let's look at an example:

Example 1: Given the equation of a function $y=f(x)$.

$f(x)=\frac{1}{2}x^2+1$ on $[0,2]$ using 4 equal subintervals.

First, we find the subinterval width:

$$\Delta x=\frac{2-0}{4}=\frac{1}{2}$$

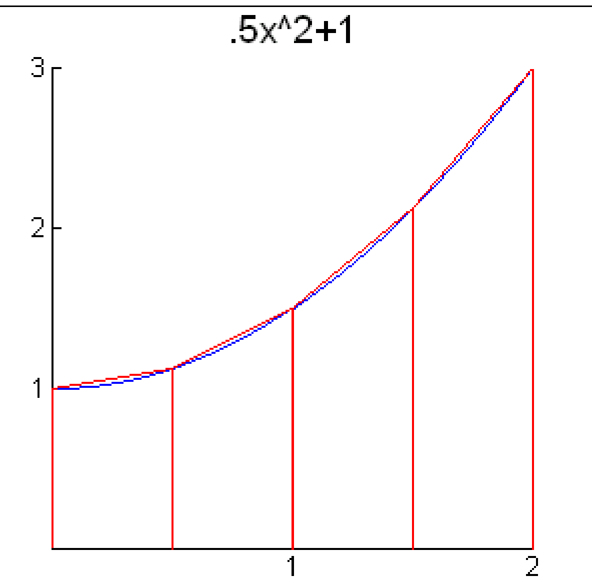
Now we plug into the formula:

$$A=\underbrace{\frac{1}{2}}_{formula}\underbrace{\left(\frac{1}{2}\right)}_{\Delta x}\left[f(0)+2f\left(\frac{1}{2}\right)+2f(1)+2f\left(\frac{3}{2}\right)+f(2)\right]$$

$$A=\frac{1}{4}\left[1+(2)\frac{9}{8}+(2)\frac{3}{2}+(2)\frac{17}{8}+3\right]$$

$$A=\frac{1}{4}\left[1+\frac{9}{4}+3+\frac{17}{4}+3\right]=\frac{1}{4}\left[\frac{4+9+12+17+12}{4}\right]$$

$$A=\frac{1}{4}\left[\frac{54}{4}\right]=\frac{54}{16}=3.375$$



This Area is a little bit MORE than the actual area since the graph is concave up. The actual area is $\frac{10}{3} = 3.333$ as shown at the right.

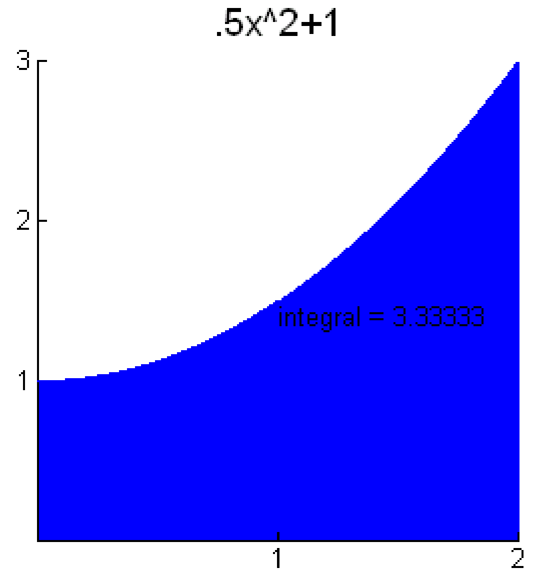
We can calculate the error now by our method of Trapezodial areas.

$$Error = A_{trapezoid} - A_{actual}$$

$$Error = \frac{54}{16} - \frac{10}{3} = \frac{162-160}{48} = \frac{2}{48} = \frac{1}{24} \approx 0.042$$

This is close enough for my wife, so it should be close enough for anybody.

Notice I used the fractions, rather than the decimal approximations. Why?

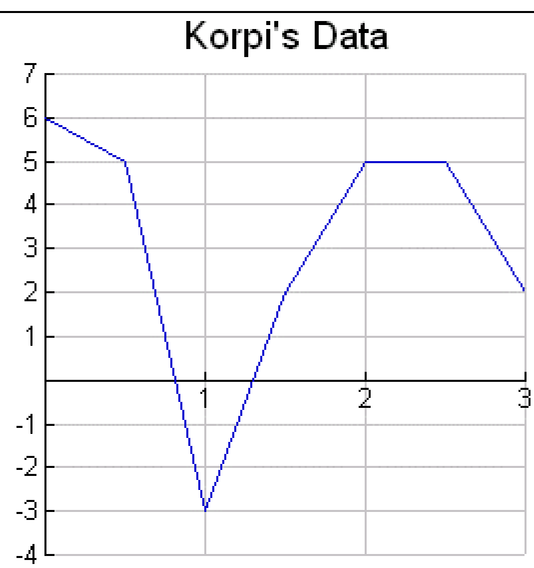


What if we aren't given a function, but rather some data? Super Question

Example 2: From Data Points

I was out collecting data again the other day. You would think it was one of my favorite hobbies, however, you'd be incorrect. Anyway, I was taking my data at ½ unit intervals. I came up with the following information.

X	***Y***
0	6
½	5
1	-3
3/2	2
2	5
5/2	5
3	2



Finding a function here would be a waste of time. I can find the area using trapezoids. Here it goes.

$$A = \frac{1}{2}\left(\frac{1}{2}\right)[6 + 2*5 + 2*-3 + 2*2 + 2*5 + 2*5 + 2]$$
$$= \frac{1}{4}[6 + 10 - 6 + 4 + 10 + 10 + 2]$$
$$= \frac{1}{4}[36] = 9$$

Remember, this means I had an accumulation of 9 units over the interval from zero to three.

We're done. Here's one last image of area approximation using trapezoids. Can you make a problem that fits the image? Let me know