5.1 The Natural Logarithmic Function and Differentiation and the Number *e*

TOOTLIFTST:

- Knowledge of derivatives of basic functions, including power, exponential, logarithmic, trigonometric, and inverse trigonometric functions.

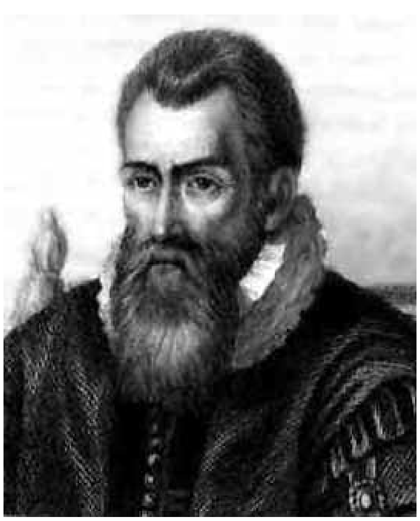
From the last section, we ran into an integral that we could not find using the power rule. If we didn't run into one, then, here, run into this one, that is, try to evaluate it:

$$\int \frac{2x}{x^2+1}\,dx$$

If we think of the "inside" function as x^2+1, in which case, it would be "inside" the function $(something)^1$, its derivative, $2x$, is precisely what is on the "outside, so there is no need for a correction. Our guess then involves the "outer" function, and becomes:

$$\frac{\left(x^2+1\right)^0}{0}+C$$

Hmmmmm. Something is wrong here. We cannot divide by zero (although Newton's method of Fluxions is based up it.) In mathematics, when something like this happens, we define a new function that works. Clever, huh? This function is called the **Natural Logarithmic Function.**



Logarithmic Functions have been around since the time of John Napier, who "discovered" them. The Natural log has, as its base, *e*, which is approximately 2.718. . . Now, unlike the logarithms used today, Napier's logarithms are not really to any base although in our present terminology it is not unreasonable (but perhaps a little misleading) to say that they are to base $1/e$. Certainly they involve a constant 10^7 which arose from the construction in a way that we will now explain. Napier did not think of logarithms in an algebraic way, in fact algebra was not well enough developed in Napier's time to make this a realistic approach. Rather he thought by dynamical analogy. Consider two lines *AB* of fixed length and *A'X* of infinite length. Points *C* and *C'* begin moving simultaneously to the right, starting at *A* and *A'* respectively with the same initial velocity; *C'* moves with uniform velocity and *C* with a velocity which is equal to the distance *CB*. Napier defined *A'C'* (= *y*) as the logarithm of *BC* (= *x*), that is

$$y = \text{Nap.log}\, x.$$

Napier chose the length *AB* to be 10^7, based on the fact that the best tables of sines available to him were given to seven decimal places and he thought of the argument *x* as being of the form $10^2.\sin X$.

Anyway, here is the definition of The Natural Logarithmic function:

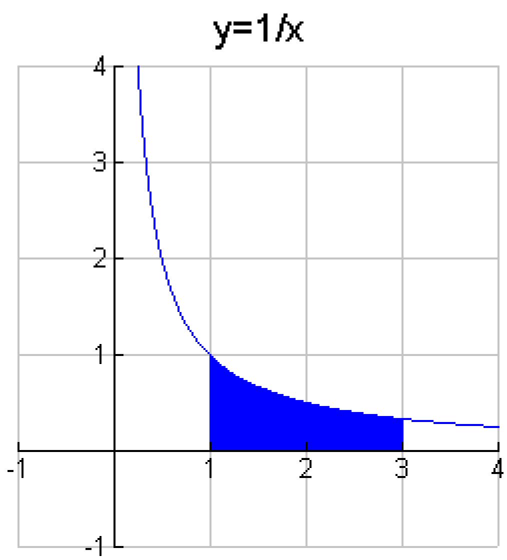
Definition of the Natural Logarithmic Function

The natural logarithmic function is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt\,,\ 0 < x$$

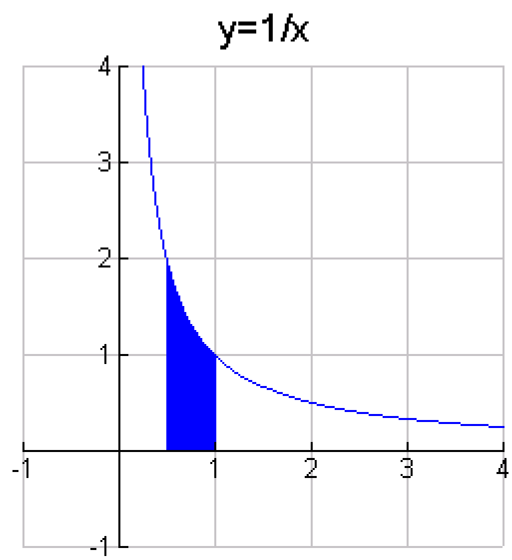
The domain of the natural logarithmic function is the set of all positive real numbers

So this means that it is defined to be the area under the curve of $1/x$ from one to any other number to the right of zero (remember, there is a Vertical Asymptote at $x = 0$)



This is $\ln 3 = \int_1^3 \frac{1}{x}\,dx > 0$

So, if $x > 1$, $\ln x > 0$

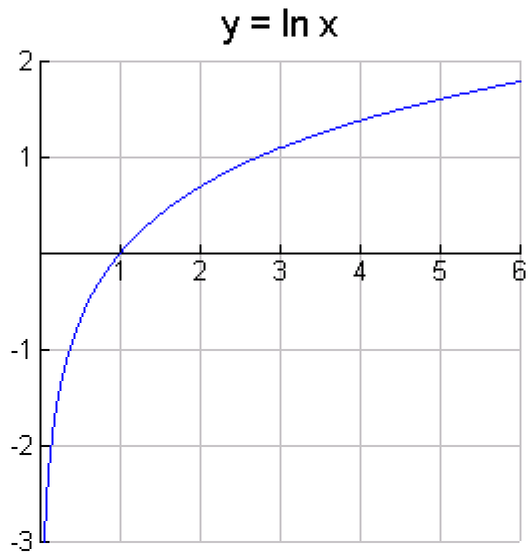


This is $\ln\frac{1}{2} = \int_1^{\frac{1}{2}} \frac{1}{x}\,dx < 0$

	So, if $0 < x < 1,\ \ln x < 0$

What is $\ln 1$?

Notice that the areas, for increasingly larger values of x, are increasing, but less, and less fast. The graph of $y = \ln x$ can be thought of as the plot of all of the points. Think about how these values look on the graph, then look below.



This function has the following properties:

1. The domain is $(0, \infty)$ and the range is $(-\infty, \infty)$.
2. The function is continuous, monotonic increasing, and one-to-one.
3. The graph is concave downward.
4. It is graph in blue in this example

Let's review some cool properties of logs, namely, they are made out of wood . . . but seriously, we'll look at natural logs (the same properties hold for logs of any base, we are just using log base e)

Logarithmic Properties
If a and b are positive numbers and n is rational, then the following properties are true.
1. $\ln(1) = 0$ (Do you know why graphically from the definition?)
2. $\ln(ab) = \ln a + \ln b$ (This expands a product into a sum—VERY NICE!!)

3. $\ln(a^n) = n\ln a$ (This eliminates exponents and treats them as scalar multiples—Ooh!)

4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$ (Just as nice as #2)

5. $\ln e = 1$ (This is partly the definition of e. More to come)

6. $\ln e^x = x$

7. $e^{\ln x} = x$

OK, we've been talking about e very intimately with the natural log, naturally, I think we should speak a little about it. We already know that

$$\ln x = \log_e x$$

But . . .

we also know that the number e first comes into mathematics in a very minor way. This was in 1618 when, in an appendix to Napier's work on logarithms, a table appeared giving the natural logarithms of various numbers. However, that these were logarithms to base e was not recognized since the base to which logarithms are computed did not arise in the way that logarithms were thought about at this time. Although we now think of logarithms as the exponents to which one must raise the base to get the required number, this is a modern way of thinking. We will come back to this point later in this essay. This table in the appendix, although carrying no author's name, was almost certainly written by Oughtred. A few years later, in 1624, again e almost made it into the mathematical literature, but not quite. In that year Briggs gave a numerical approximation to the base 10 logarithm of e but did not mention e itself in his work.

The next possible occurrence of e is again dubious. In 1647 Saint-Vincent computed the area under a rectangular hyperbola ($xy = 1$ which equals $y = 1/x$). Whether he recognized the connection with logarithms is open to debate, and even if he did there was little reason for him to come across the number e explicitly. Certainly by 1661 Huygens understood the relation between the rectangular hyperbola and the logarithm. He examined explicitly the relation between the area under the rectangular hyperbola $yx = 1$ and the logarithm. Of course, the number e is such that the area under the rectangular hyperbola from 1 to e is equal to 1. This is the property that makes e the base of natural logarithms, but this was not understood by mathematicians at this time, although they were slowly approaching such an understanding.

Huygens made another advance in 1661. He defined a curve which he calls "logarithmic" but in our terminology we would refer to it as an exponential curve, having the form $y = ka^x$. Again out of this comes the logarithm to base 10 of e, which Huygens calculated to 17 decimal places. However, it appears as the calculation of a constant in his work and is not recognized as the logarithm of a number (so again it is a close call but e remains unrecognized).

Further work on logarithms followed which still does not see the number e appear as such, but the work does contribute to the development of logarithms. In 1668 Nicolaus Mercator published *Logarithmotechnia* which contains the series expansion of log(1+x). In this work Mercator uses the term "natural logarithm" for the first time for logarithms to base e. The number e itself again fails to appear as such and again remains elusively just round the corner.

Perhaps surprisingly, since this work on logarithms had come so close to recognizing the number e, when e

is first "discovered" it is not through the notion of logarithm at all but rather through a study of compound interest. In 1683 Jacob Bernoulli looked at the problem of compound interest and, in examining continuous compound interest, he tried to find the limit of $(1 + {}^{1}/_{n})^n$ as n tends to infinity. He used the binomial theorem to show that the limit had to lie between 2 and 3 so we could consider this to be the first approximation found to e. Also if we accept this as a definition of e, it is the first time that a number was defined by a limiting process. He certainly did not recognize any connection between his work and that on logarithms.

That was taken from the MacTutor Archives. I hope that shed the light on e. Basically, from the formal definition of $\ln x$, to generate an area under the curve of $y = \frac{1}{x}$ (a type of square hyperbola) of exactly ONE UNIT, we must integrate from 1 to 2.718 . . . or ***e***.

That is quite fascinating, in light of the numerous other places e shows up. (like in the binomial expansion for accumulated interest and in the alphabet.)

From the properties above, it a no coincidence that e and $\ln x$ seem to "undo" each other. They are actually inverse operations of each other. We will discuss this more in a future lesson.

Let's see the power of these rules:

1. $\ln\sqrt{x^2+1} = \ln\left(x^2+1\right)^{1/2} = \frac{1}{2}\ln\left(x^2+1\right)$ (You cannot distribute the $\ln$ to each term because it is an *OPERATION*!!)
2. $\ln\dfrac{x^3\sqrt[3]{x^2-2x}}{x\sin x} = 3\ln x + \frac{1}{3}\ln\left(x^2-2x\right) - \ln x - \ln\sin x$ (If a factor is in the numerator, it is ADDED. If a factor is in the denominator, it is SUBTRACTED. All exponents come out front. Don't forget to give each term its own $\ln$.)
3. $5\ln\left(x^3-1\right) - 2\ln x + \ln(x+1) - 2\ln(\sin x) = \ln\dfrac{\left(x^3-1\right)^5(x+1)}{x^2\sin^2 x}$ (The rules work in reverse, too, to condense expressions.)

So that's great. The rules help us expand a product or quotient into a sum and to condense a sum into a product or quotient. Why is that so useful? The answer to that question is soon to come, but first, let's visit the derivative of $\ln x$.

So, what ***is*** the derivative of $\ln x$? You probably already have this one figured out from the definition, but let's state the obvious here:

Since $\ln x = \int_1^x \frac{1}{t}\,dt$ when we take the derivative with respect to x of both sides, we can use the 2nd Fundamental Theorem of Calculus to get the following:

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Chances are, throughout the remainder of the year, I will ask you to take the derivative of expressions that are a little more sophisticated than just the derivative of the natural log of "plain ol' *x*." I will ask you to find the derivative of the natural log of a ***function*** of *x*. We can call this function of *x*, *u*. In this case, the faithful chain rule comes back to give us a hand.

$$\frac{d}{dx}\ln u = \left(\frac{1}{u}\right)(du) \text{ or } \frac{du}{u}$$

This simply means, if we are taking the derivative of the natural log of "something", it will be equal to the derivative of that "something" over the "something" itself. Rewriting any complex product or quotient into multiple terms by using the above rules, makes differentiating with natural log a piece of cake and leaves you wanting more cake.

Examples:

$$\frac{d}{dx}\ln(x^2+2x-5) = \frac{\frac{d}{dx}\left(x^2+2x-5\right)}{x^2+2x-5} = \frac{2x+2}{x^2+2x-5}$$

$$\frac{d}{dx}\ln x\sqrt{x^2-7} = \frac{d}{dx}\left[\ln x + \frac{1}{2}\ln\left(x^2-7\right)\right] = \frac{1}{x}+\frac{1}{2}\left(\frac{2x}{x^2-7}\right) = \frac{1}{x}+\frac{x}{x^2-7}$$

$$\frac{d}{dx}\ln\frac{x^3\sqrt{x^2-x}}{\sin^2 2x} = \frac{d}{dx}\left[3\ln x + \frac{1}{2}\ln\left(x^2-x\right) - 2\ln(\sin 2x)\right] = \frac{3}{x}+\frac{2x-1}{2\left(x^2-x\right)}-\frac{4\cos 2x}{\sin 2x}$$

This new rule can be combined with other, already established rules, such as the Product Rule or the Chain Rule:

Examples:

$$\frac{d}{dx}x\ln x = (1)(\ln x)+(x)\left(\frac{1}{x}\right) = \ln x + 1$$

$$\frac{d}{dx}\ln\left(\ln\left(2x^2+5x-2\right)\right) = \frac{\frac{d}{dx}\ln\left(2x^2+5x-2\right)}{\ln(2x^2+5x-2)} = \frac{\frac{4x+5}{2x^2+5x-2}}{\ln\left(2x^2+5x-2\right)} = \frac{4x+5}{\left(2x^2+5x-2\right)\left(\ln\left(2x^2+5x-2\right)\right)}$$

$$\frac{d}{dx}\ln^4\left(\tan^2 2x\right) = \frac{d}{dx}\left[\ln\left(\tan(2x)\right)^2\right]^4 = 4\left[\ln(\tan(2x))^2\right]^3\left(\frac{2(\tan(2x))(\sec^2(2x))(2)}{\tan^2(2x)}\right) = \frac{16\sec^2 2x\ln\left(\tan^2 2x\right)}{\tan 2x}$$

Notice on the above example, I didn't pull the 2 exponent from the tangent to the front. This would have made the derivative a bit simpler. But I did it to make a point. It is not necessary to use the expansion rules of logs, but I sure makes the job easier.

So now we see how the rules of logs help us simplify the derivative process when there is a log already in the problem. But what if there isn't? Well, then, we can simply put one there . . . almost.

Take this for example:

$$y = \frac{2x^3\left(x^2-5\right)}{\sqrt{1-x^4}}$$

To differentiate this the conventional way, since there are no natural logarithms here, would require a combination of the quotient rule, the product rule, and the chain rule—in the correct sequence. Now, this is something each of you is more than capable of doing, BUT why show off when we don't have to. That might be considered haughty.

Instead, Let's use the rules of logs *by introducing one to **both** sides*! How clever.

$$\ln y = \ln \frac{2x^3\left(x^2-5\right)}{\sqrt{1-x^4}}$$ We have now changed the problem, but not the solutions.

Continuing with the expansion on the right side, we want to rewrite BOTH sides as we work down so that we won't forget to perform an important step at the end.

$$\ln y = \ln 2 = 3\ln x + \ln(x^2-5) - \frac{1}{2}\ln(1-x^4)$$

Now we can take the derivative with respect to x of both sides.

$$\frac{d}{dx}\left[\ln y = \ln 2 + 3\ln x + \ln(x^2-5) - \frac{1}{2}\ln(1-x^4)\right]$$

$$\frac{y'}{y} = 0 + \frac{3}{x} + \frac{2x}{x^2-5} - \frac{1}{2}\left(\frac{-4x^3}{1-x^4}\right)$$

The last thing we need to do is solve for y' which is the same as $\frac{dy}{dx}$. All we need to do is multiply both sides of the equation by y. This is the most forgotten step. But what is y really equal to? In this case, we were told at the beginning that $y = \frac{2x^3\left(x^2-5\right)}{\sqrt{1-x^4}}$ so the final equation becomes

$$y' = \left[\frac{3}{x} + \frac{2x}{x^2-5} + \frac{2x^3}{1-x^4}\right]\left[\frac{2x^3\left(x^2-5\right)}{\sqrt{1-x^4}}\right]$$

And we are done. No need to combine the two.

What we just did is so important, and is such an important tool to help us find derivatives of complicated products and quotients that it has its own name: Logarithmic Differentiation. Once you get really adept and proficient at using this method, you can refer to it by its nickname: LOG DIFF.

One final interesting note regarding absolute value: If we wanted to take the derivative of the natural log of the absolute value of a function, I claim that we can treat the problem as if there were no absolute value signs to begin with. Do you believe my claim? (Warning: Not all my claims are true. I once claimed that my Grandfather passed away at the tender age of two. This, of course, was totally false. He actually died at age one.)

Here's what I'm talking about:

Let $y = \ln|u|$ where u is a function of x. Then $y' = \frac{u'}{u}$.

Proof:
Case 1: If $u > 0$, then we don't need the absolute value at all, and the previous rule holds, but . . .

Case 2: If $u < 0$ then $|u| = -u$, so the problem becomes $y = \ln(-u)$, in which case the derivative would be $y' = \frac{-u'}{-u} = \frac{u'}{u}$, since the negatives divide out.

Example:

$$\frac{d}{dx}\ln|\cos x| = \frac{-\sin x}{\cos x} - \tan x$$

And this concludes the lesson on the derivation of the derivative of the natural log function.