5.2—The Integral of the Natural Log

TOOTLIFTST:

Now that we know how to go forward with the natural log (that is, take the derivative of it) we can now go forward to the next step, going backward (that is, take the integral.)

Here's the theorem:

The Log Rule of Integration

Let u be a differentiable function of x.

1. $\int \frac{1}{x} dx = \ln|x| + C$
2. $\int \frac{1}{u} du = \ln|u| + C$

Now, because $du = u'dx$, we can modify #2 to become:

3. $\int \frac{u'}{u} dx = \ln|u| + C$

So what's up with the absolute value sign? Why do we need it? Remember, the domain of $\ln x$ is all $x > 0$. You cannot take the natural log of negative numbers (or zero). BUT, the original function in the integral may take on negative values. To be sure that none of these find their way into the antiderivative, we add the absolute value signs to be safe.

Here are some quick examples:

Example 1:

$$\int \frac{2}{x} dx = 2\int \frac{1}{x} dx = 2\ln|x| + C$$

Here's an important Korpi rule. When it is possible to eliminate the absolute value sign from the problem, I want you to do it. I consider it "more simplified" although it would still be technically correct to leave them in there.

In the above problem, using the rules of logs, we can eliminate the absolute value sign by bringing the scalar multiple 2 up as an exponent, obtaining

$$\ln(x^2) + C$$

This is now a good place to discuss notation etiquette. If the parenthesis were left off above, $\ln x^2 + C$, we would still assume we were squaring only the x before we take the natural log. If we intended to square the entire quantity, we would have to write it as $(\ln x)^2 + C$ or the more preferable shorthanded form $\ln^2 x + C$.
Let's look at another example and crescendo in difficulty up to darn near impossible.

Example 2.

Evaluate $\int \frac{dx}{4x-1}$

First, we rewrite: $\int (4x-1)^{-1}\,dx$

Now you are probably used to looking for "inside" functions whose derivative is hopefully on the "outside." This is what the last section was about. In this case, the $4x-1$ is "inside" something to the negative first. It's derivative is 4, which is not another factor on the "outside," so you have to make a correction of $1/4$. If you proceed with the power rule, you obtain $\left(\frac{1}{4}\right)\left(\frac{(4x-1)^0}{0}\right)+C$. RUGHHHH ROOOOOGH.

This is precisely the situation in which we call on the services of the natural log: when our "inside" function is either in the denominator to the first power, or in the numerator to the negative first power. *In such a singular case*, the power rule will not work.

Instead, the answer becomes $\left(\frac{1}{4}\right)\ln|4x-1|+C$

In this problem, we are again able to eliminate the absolute value sign since the fourth root is an EVEN root.

The final, simplified answer is then $\ln\sqrt[4]{4x-1}+C$

If the original problem would have had an exponent on the $4x-1$ term other than negative one, the power rule would have worked.

Example 3: A definite Integral

Evaluate $\int_0^2 \frac{x}{x^2+1}\,dx$

Notice that x^2+1 is in the denominator and is "inside" something to the first power, $(x^2+1)^1$, AND notice that its derivative, $2x$, is only off by a 2. This is another job for natural log.

$$= \frac{1}{2}\ln\left(x^2+1\right)\Big|_0^2 = \frac{1}{2}\left[\ln 5 - \ln 1\right] = \frac{1}{2}\left[\ln 5 - 0\right] = \frac{1}{2}\ln 5 \text{ or } \ln\sqrt{5} \approx 0.805$$

Notice two things. First, we don't need absolute value brackets. Why? And Second, we could have evaluated it a bit differently in the third step.

$\frac{1}{2}\left[\ln 5 - \ln 1\right] = \frac{1}{2}\ln\left(\frac{5}{1}\right) = \frac{1}{2}\ln 5$. Using the rules of condensing, we get the same answer.

How do YOU see the problem?

Example 4: Quick Recognition: Getting Good at it

Quickly, as fast as you can without sacrificing accuracy, find the following integrals by pattern recognition.

1. $\int \frac{3x^2+1}{x^3+x}dx$
2. $\int \frac{\sec^2 x}{\tan x}dx$
3. $\int \frac{x+1}{x^2+2x}dx$
4. $\int \frac{1}{3x+2}dx$

If you want to see if you're correct, check your textbook on page 328.

Example 5: A new clever trick

Evaluate $\int \frac{x^2+x+1}{x^2+1}dx$

None our ticks will work here. The fact that we will use the log rule is not immediately obvious. The derivative of either the top of bottom function is not the other factor. The thing to do when you are in this situation, with a rational function where the degree of the numerator is $\geq$ the degree of the denominator is . . . . . .

LONG DIVISION

$$\begin{array}{r} 1 \\ x^2+1 \overline{\smash{\big)}\, x^2+x+1} \\ \underline{x^2+1} \\ x \end{array}$$

This means that the original integral can be rewritten as One plus the remainder over the divisor.

$$\int 1 + \frac{x}{x^2+1}dx$$

This integral is now two smaller problems:

$$\int 1dx + \int x\left(x^2+1\right)^{-1} dx$$

$$= x + \frac{1}{2}\ln\left(x^2+1\right) \text{ or } x + \ln\sqrt{x^2+1}$$ (absolute values not needed on either one)

Example 6: u-substitution, hidden log rule

Evaluate $\int \frac{2x}{(x+1)^2}dx$

The derivative of the "inside" function is only 1. We are off by a $2x$. All you know how to do at this point is u-substitution. The problem definitely doesn't appear to involve natural logs . . . yet.

Let $u = x+1$. $du = dx$ and $x = u-1$.
Substituting and pulling the 2 out front, we get

$$2\int \frac{u-1}{u^2}du = 2\int \frac{u}{u^2} - \frac{1}{u^2}du = 2\int u^{-1} - u^{-2}du = 2\ln|u| + \frac{1}{u} + C$$

Plugging the original function back in for u (No, do it yourself), you get

$$2\ln|x+1| + \frac{1}{x+1} + C$$

Notice in the second step above, we were able to split the fraction up into smaller fractions. When this is possible, it is often beneficial.

The last two examples required rewriting a disguised integrand so that it fit one or more of our basic formula rules. We will be increasing our arsenal of integration formulas, each fitting a different pattern. It is very important that you get accustomed to seeing these patterns in order to master integration. This is the hardest part about "going backwards": there are no straightforward methods you can blindly apply ever time. It is more of an evolved problem-solving skill that can only be developed through a keen eye, and a tenacious and persistent effort.

So far, we have the following *integration techniques*:

1. Look to see if the integrand is a known derivative of another function.
   Ex) $\int \sec x \tan x dx = \sec x + C$

2. Can the power rule be used immediately, or can the function be rewritten so that it

can?
Ex) $\int 3x^2 + 2x - 1dx$
Ex) $\int \frac{3x^2 + 2x}{x} dx = \int 3x + 2dx$

3. If the integrand is a product or quotient of two functions, can we use the power rule and pattern recognition or *u*-substitution?
   Ex) $\int \tan^4 x \sec^2 x dx$

4. If the pattern recognition doesn't work, is it because we need to use the log rule?
   Ex) $\int \frac{2x}{x^2 + 1} dx$
   Ex) $\int \frac{1}{x \ln x} dx$

5. Can the integrand be manipulated to fit a know formula? Try using trig identities, multiplying by clever forms of one, adding clever forms of zero, long division, splitting a fraction up into two or more . . . Be creative.

Here are a couple more examples that are so similar they could be cousins, but yet so different they are probably very distant cousins.

Example 7:

Evaluate $\int \frac{1}{x \ln x} dx$. This one is helpful if it is rewritten as $\int \left(\frac{1}{x}\right)\left(\frac{1}{\ln x}\right) dx$.
Of the two possible choices for the "inside" function, the derivative of $\ln x$ is $1/x$, which is the remaining factor on the outside. The "inside" function resides in the denominator and is raised to the first power, $\left(\frac{1}{(\ln x)^1}\right)$, which make it an ideal candidate for the log rule. Our correct guess then becomes

$\ln(\ln x) + C$

So we can use the log rule on ANY inside function, including itself.

Here's this one's distant cousin

Example 8:

Evaluate $\int \frac{\ln^2 x}{x} dx$

Rewriting, we get $\int \left(\frac{1}{x}\right)(\ln x)^2 dx$

Our "inside" function is again $\ln x$ (hence the similarity) but it is now in the numerator to the second power, not in the denominator to the first. This makes it a power rule, rather than a log rule.

Our correct guess is then

$$\frac{(\ln x)^3}{3} + C = \frac{1}{3}\ln^3 x$$

Done.

We will now conclude with the derivation of two of six integrals you will have to ***memorize***. They each require a clever rewriting of the integrand.

Example 9:

Evaluate $\int \tan x dx$

If you proceed down the list of strategies we have, none of them work. What we can do is tap into our precalculus resources and pull out a Ration Identity.

$\int \tan x dx = \int \frac{\sin x}{\cos x} dx$ Now we have two options for the "inside" function. The derivative of $\sin x$ IS $\cos x$, but it would have to be $1/\cos x$ or $\sec x$, since the cosine is in the denominator. The derivative of $\cos x$ is $-\sin x$. Since sine is in the numerator, we are only off by a negative sign, and because our "inside" function now is in the denominator to the first power, $(\cos x)^1$, our guess involves the natural log, naturally.

$$\int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + C$$

Bringing the negative up as an exponent, we get an equivalent form

$$\ln\left|(\cos x)^{-1}\right| + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C$$

Now, if you ever forget this—remember, you are to memorize it—you can always derive it quickly. The next one, however, will be more difficult to derive should you forget it. It requires a more clever trick.

Example 10: Multiplying by a fraction equivalent to 1.

Evaluate $\int \sec x dx$

This one seems harmless enough, but it is not a know derivative of another trig function, and rewriting this one as $1/\cos x$ doesn't help like it did above. Instead, we use the fact that $\sec x$ and $\tan x$ are associated through there derivatives and decide to do the following

$$\int \sec x dx = \int \sec x dx \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$

So how does that help? It appears we have severely complicated the problem. But as Churchill said, "Out of intense complexities, intense simplicities emerge," so is the case here. Notice that the derivative of the denominator is precisely the numerator (remember, addition is commutative and associative so it is no problem to invert the terms to get what we need.) With our "inside" function now being in the denominator to the first power, the power rule dictates the answer to be

$$\ln\left|\sec x + \tan x\right| + C$$

Here is a summary of the six Integrals of trig functions you need to know by heart (2 of them you already know.

$\int \sin u du = -\cos u + C$	$\int \cos u du = \sin u + C$
$\int \tan u du = -\ln\left|\cos u\right| + C$	$\int \cot u du = \ln\left|\sin u\right| + C$
$\int \sec u du = \ln\left|\sec u + \tan u\right| + C$	$\int \csc u du = -\ln\left|\csc u + \cot u\right| + C$

Remark: As we did above for $\int \tan u du$, each of these have equivalent forms that can be obtained by using trig identities and properties of logs.

And finally, the encore Example, using trig identities.

$\int_0^{\pi/4} \sqrt{1+\tan^2 x} dx$ Remembering the Pythagorean Identities and the Unit Circle is a must!!!!

$$= \int_0^{\pi/4} \sqrt{\sec^2 x} dx = \int_0^{\pi/4} \sec x dx = \ln\left|\sec x + \tan x\right|\Big|_0^{\pi/4} = \ln\left(\sqrt{2}+1\right) - \ln 1 \approx 0.881$$