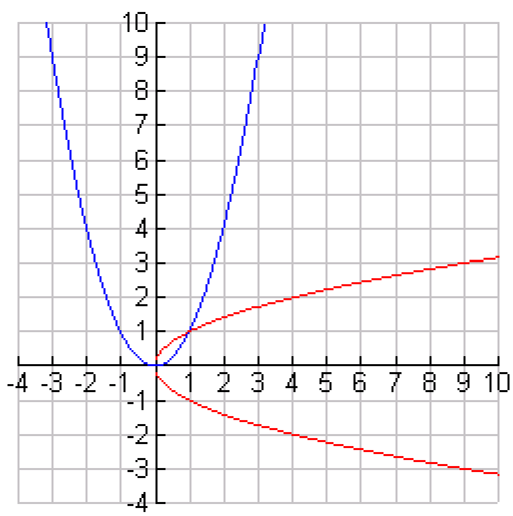
5.3 Inverse Functions

TOOTLIFTST:

- Knowledge of derivatives of basic functions, including power, exponential, logarithmic, trigonometric, and inverse trigonometric functions.

Now each of you should be familiar with inverses from your previous mathematical studies. The inverse function of f, denoted as f^{-1}, can be thought of as a function that "undoes" the work of the original function. It is like the decoder for the code, the anti-venom to the poison. Not every function has an inverse that is also a function, like $f(x)=x^2$. The opposite of squaring an *x* is taking the square root of it, so $f^{-1}(x)=\sqrt{x}$, but we'd have to graph $g(x)=-\sqrt{x}$ to achieve the lower half to the sideways parabola.



The blue graph is $f(x)=x^2$. The red graph is the graph of its inverse $y=\pm\sqrt{x}$. Notice the red graph does not pass the vertical line test, and is therefore NOT a function. We could restrict the domain of $f(x)=x^2$ to only positive values of x, only then will the inverse be a function, the upper red portion.

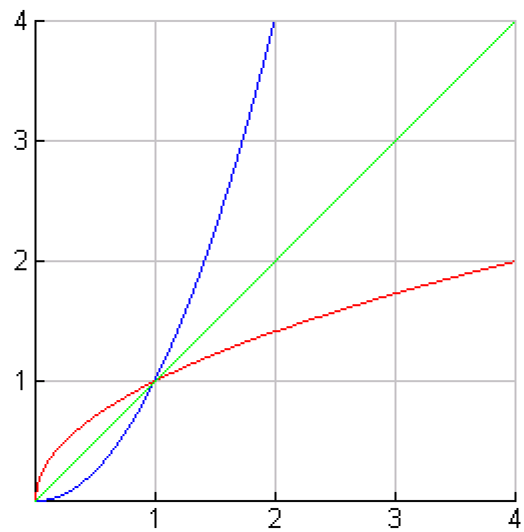
In this chapter, we will be concerned with functions whose inverses are automatically functions, without having to restrict the domains. We will refer to inverse functions simply as inverses.

How do we find inverses, and how can we tell if they will be functions?

Well, algebraically, recall that inverses are found by switching *x* and *y*, then solving for *y*.
Example:
If $f(x)=2x^3-1$, find its inverse

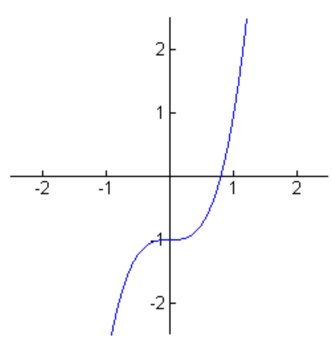
$y=2x^3-1$	First, we replace $f(x)$ with *y*.
$x=2y^3-1$	Switch *x* and *y*.
$y=\sqrt[3]{\frac{x+1}{2}}$	Solve for *y*.
$f^{-1}(x)=\sqrt[3]{\frac{x+1}{2}}$	Replace with inverse function notation

They key step in what we just did was the interchanging the x and y coordinates. This gives us a hint of how to find or generate inverses graphically. Think of a function that is identical when the x and y are switched. Do you have it yet? It is $y = x$. If you are not convinced, try it yourself. You should be done now. It works. Now, because this is the only such function that is its own inverse, its graph becomes the “mirror” across which a function will reflect to generate its inverse. Let’s go back to the example above. When the line $y = x$ is graph on the same axis, you can see (if you turn you head to your right $45°$) that they are reflections about that line.



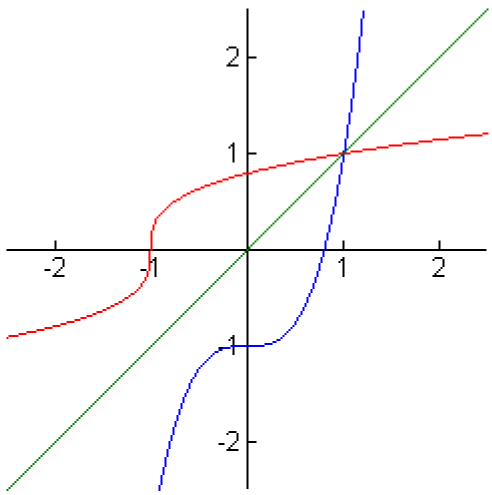
The interesting thing to note here is that <u>everything</u> x and y switch: Points, axes, Domains, Ranges, etc. A point (a,b) is on the graph of f <u>if and only if</u> (b,a) is a point on f^{-1}.

So now we know how to find the equation and graph of an inverse given a function, but how can we tell just from the function that the inverse will also be a function? Let’s examine the graph of $f(x) = 2x^3 - 1$



Because it is a function, it passes the vertical line test, which means that for each x, there is no more than a single y-value associated with it. This means we want the Inverse to pass this same Vertical line test. But realize that a vertical line, when reflected across the line $y = x$ will become a Horizontal Line!! This is precisely why the x and y axis (domains and ranges) switch with inverses.

SO, if the original function passes both a vertical AND horizontal line test, the inverse should also be a function, RIGHT!! Such functions are called **<u>ONE-TO-ONE</u>**. Our graph above passes both test, hence, is one-to-one, hence has an inverse (which we already found to be $f^{-1}(x) = \sqrt[3]{\frac{x+1}{2}}$. Here are the two of them graphed together.



When we compose a function with its inverse in either order, we end up right back where we started, with x. This property is summarized here

If f and g are inverses, then $f(g(x)) = g(f(x)) = x$

Verify this for the two functions above.

Now recall the definition of **<u>strictly monotonic</u>**: A function is strictly monotonic if it is increasing or decreasing over its entire domain. A function that is one-to-one must also be monotonic (except horizontal lines, which are not increasing nor decreasing), but not all monotonic functions are one-to-one. Can you find an example? (Hint: Think tangent of x.) These would make good true or false questions on test.

To determine one-to-oneness, it is customary to view the graph and perform the horizontal and vertical line test. It is less obvious from the equation alone.

Let's get calculus involved finally, and compare the slopes of these graphs. The ordered pair $\left(\frac{1}{2},-\frac{3}{4}\right)$ is point on the function $f(x)=2x^3-1$ (the blue graph above) which means that $\left(-\frac{3}{4},\frac{1}{2}\right)$ is a point the inverse graph $f^{-1}(x)=\sqrt[3]{\frac{x+1}{2}}$, the red graph. Let's compare the slopes of the tangent lines of these points on their respective graphs.

$$f'(x)=6x^2,\ f'\left(\frac{1}{2}\right)=6\left(\frac{1}{2}\right)^2=\frac{3}{2}$$

and

$$\left(f^{-1}\right)'(x)=\frac{1}{3}\left(\frac{x+1}{2}\right)^{-2/3}\left(\frac{1}{2}\right),\ \left(f^{-1}\right)'\left(-\frac{3}{4}\right)=\left(\frac{1}{6}\right)\left(\frac{1}{8}\right)^{-2/3}=\frac{2}{3}$$

The slopes of the tangent lines at the inverse points are RECIPROCALS of each other. This is true for **all slopes** of inverses. We can summarize this by the following equation.

The Derivative of an Inverse Function

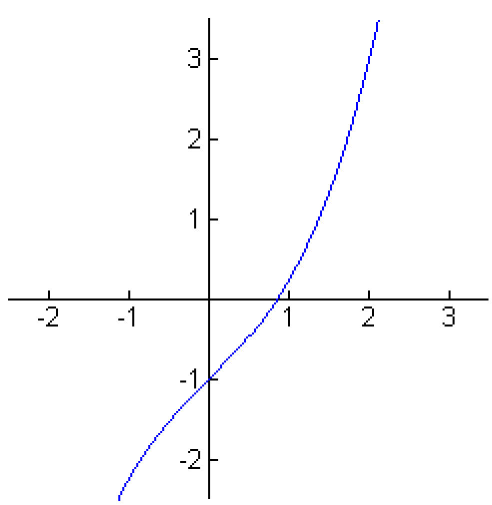
Let *f* be a function that is differentiable on an inverval *I*. If *f* possesses an inverse function *g*, such that $g=f^{-1}$, then *g* is differentiable at any *x* for which $f'(f(x))\neq 0$. Moreover,

$$g'(x)=\frac{1}{f'(g(x))},\quad f'(g(x))\neq 0.$$

Very often, we will be interested in knowing what the slope of the tangent line is of the inverse function at a specific point. If we can find the equation of the inverse function, this is as difficult as finding the inverse, taking the derivative, and plugging in the *x*-value.

Now, some functions have inverses, but the equation of the inverse is not easily obtainable from the original function. When this happens, we need to think of the inverse graphically or in terms of the original function, for example

$$f(x)=\frac{1}{4}x^3+x-1$$



Since there are multiple *x* terms of different degrees, it will be difficult solving for *y* when the variables are switched. We do know, however, that the inverse exists because of the graph of f (to the left.)

But how can we find the slope of the inverse function at a specific point without the equation? Well, we can use the equation above. Let's try it. Let's say we want to know what $\left(f^{-1}\right)'(3)$, that is , the slope of the tangent line of the inverse at $x=3$. To do this, we need to know what $f^{-1}(3)$, that is, the solution to $f(x)=3$. To find this, we set the original function equal to 3, and solve for x on the calculator by finding the x-intercept.

$$f(x)=\frac{1}{4}x^3+x-1=3$$
$$\frac{1}{4}x^3+x-4=0 \Rightarrow x=2$$

This means $f(2)=3$ so $f^{-1}(3)=2$

So to find the slope of the inverse at $x=3$, all we need to know is the slope of the original function at $x=2$. We know this will be the reciprocal of the slope we want.

$$f'(x)=\frac{3}{4}x^2+1$$
$$f'(2)=\frac{3}{4}\left(2^2\right)+1=4$$

Therefore, the slope we want is actually $\frac{1}{4}$.

Formally, $\left(f^{-1}\right)'(3)=\dfrac{1}{f'\left(f^{-1}(3)\right)}=\dfrac{1}{f'(2)}=\dfrac{1}{4}$

We did all this without even knowing the inverse function!!

To summarize, to find the slope of an inverse function at a specific point:

1. Wet the original function equal to the desired value and solve for x.
2. Find the derivative of the original function and evaluate it at the value of x you found in number 1.
3. Take the reciprocal of this number to get your answer.

Graphs of Inverse functions have reciprocal slopes.

*Note: when trying to find slopes of inverse functions, it is proper to see if the function is one-to-one first, otherwise, the inverse slopes can still be found, but there might be more than one answer. See number 63, page 342.