Summary of Integration Techniques

First of all, the most important and integral factors in solving any integration problem are recognizing the pattern so that the correct integration rule can be applied. In order to do this, you must MEMORIZE the rules, and you must PRACTICE them over and over again until you become proficient at using them. It is one thing to know HOW to integrate using a particular rule; it is another thing altogether to know WHEN to use that rule. The thing that can seem so daunting, challenging, and fun about the whole process is that integrands that are only slightly different can lead to some dramatically different solutions. Please reread this paragraph.

Here's an analogy of Really knowing the rules versus recognizing them or using a chart. Think about the game of Chess. There are many rules and many strategies. You can read all the books in the world about how to be a great chess player, but you can never actually be great (or even considered a player) if you never play, and play a lot. Also, suppose you were a horrible chess player, you didn't even know the rules, and I sat you down with the great Bobby Fisher. He had his talent, which was refined over and over again by practice, practice, and more practice. You have a rulebook and strategy guide. Not only are you going to the Royal snot beat out of you, but it would take FOREVER because you would have to research your options, one move at a time because you don't have the big picture. Do you get the picture? To make a good and formidable chess mate, you need to have the rules and strategies memorized and have practiced enough, so that you can play in an engaging manner, see moves that are still three, four, or five moves away, and create new strategies and clever techniques as the situation demands. So Integration is just a game of chess! (Incidentally, I don't play chess at all I'm a checkers guy.)

Hopefully, if you work enough different types of problems, you can spot the rule almost immediately. Until you do, though, I am going to go through a sequence of things that you should try, giving an example of each. Think of it as an integral flow chart.

First: *Look to see if the problem is a polynomial or can be written as a power rule, or has a known derivative, like trig functions*

Ex) $\int 3x^3 + 4x^2 + \frac{1}{2}x + 3 = \frac{3}{4}x^4 + \frac{4}{3}x^3 + \frac{1}{4}x^2 + 3x + C$

Remember, treat scalar multiples as curious bystanders who are just along for the ride. If you just apply the POWER RULE to the variable terms, the preexisting scalar can then be combined with the new coefficient from the power rule AFTER integrating, for example:

Ex) $\int 8\sqrt{x}dx = \int 8 \underset{rewriting}{x^{1/2}} dx = 8\left(\frac{2}{3}\right)x^{3/2} + C = \frac{16}{3}x^{3/2} + C$

Or

Ex) $\int \frac{8}{\sqrt[3]{x^2}}dx = \int 8x^{\frac{-2}{3}}dx = 8(3)x^{1/3} + C = 24x^{1/3} + C$

Now in the very rare event where you'll be asked to ingrate plain ol' sine of x, remember it is a known derivative, you just need to make your correction of the unwanted scalar multiple, in this case negative one.

Ex) $\int \sin x dx = \underset{correction}{-} \cos x + C$

This known derivative is actually more helpful for the General Power Rule, when there is another factor in the integrand. We'll get to that later. For now, here are some other know trig derivatives, two that are straight up memorization, one that requires a clever rewrite first.

Ex) $\int \sec^2 x dx = \tan x + C$

Ex) $\int \tan x dx = -\ln|\cos x| + C$

Ex) $\int \frac{\tan x}{\cos x}dx = \int \sec x \tan x dx = \sec x + C$

Notice how the last one can really offer some other interesting and indirect ways of integrating, for example, using trig identities

Ex)

$$\int \frac{\tan x}{\cos x}dx = \int \frac{\frac{\sin x}{\cos x}}{\cos x}dx = \int \frac{\sin x}{\cos^2 x}dx = \int \sec^2 x \sin x dx = \int (\tan^2 x + 1)\sin x dx = \int \tan^2 x \sin x + \sin x dx = \ldots$$

This is where a little forethought and knowing the rules pays off.

But what if it isn't a known derivative and there is more that one factor either as a product or a quotient?

Second: Look to see if you can use a modified version of the above, where the "inside" function can be replaced with *u*, and the other factor is *du*.

Because the extra factor comes from the chain rule upon differentiating, the solution involves only the "outside" function that contains the "inside" function. This technique involves the general power rule, the natural log rule, trig rules, and exponential rules. It can only be used if *du* is only off by a scalar multiple. THE MAJORITY OF THE INTEGRALS YOU WILL SEE THIS YEAR CAN BE DONE THIS WAY!!!!!

Ex) $\displaystyle\int \frac{x+1}{\left(x^2+2x\right)^3}dx = \underbrace{\frac{1}{2}}_{correction}\left(\underbrace{-\frac{1}{2}}_{power\ rule}\right)\left(x^2+2x\right)^{-2}+C=-\frac{1}{4\left(x^2+2x\right)^2}+C$

Ex) $\displaystyle\int \frac{x+1}{x^2+2x}dx = \underbrace{\frac{1}{2}}_{correction}\ln\left|x^2+2x\right|+C$

(remember, the log rule is for the singular case of the general power rule when the "outside" function is in the denominator to the first power, or in the numerator to the negative first!)

Ex)

$$\int \frac{3x+3}{\sqrt{x^2+2x}}dx = 3\int \frac{x+1}{\left(x^2+2x\right)^{1/2}}dx = \underbrace{3}_{scalar}\left(\underbrace{\frac{1}{2}}_{correction}\right)\left(\underbrace{2}_{power\ rule}\right)\left(x^2+2x\right)^{1/2}+C=3\sqrt{x^2+2x}+C$$

Not every product or quotient will be from the chain rule. Don't be fooled by problems that are polynomials in disguise . . .

Ex) $\displaystyle\int x\left(x^3+2x\right)dx=\int x^4+2x^2dx=\frac{1}{5}x^5+\frac{2}{3}x^3+C$

Or

Ex)

$$\int \frac{3x^3-2x^2+x-1}{\sqrt{x}}dx=\int 3\frac{x^3}{x^{1/2}}-2\frac{x^2}{x^{1/2}}+\frac{x}{x^{1/2}}-\frac{1}{x^{1/2}}dx=\int 3x^{5/2}-2x^{3/2}+x^{1/2}-x^{-1/2}dx=\ldots$$

Involving trig:

Ex) $\displaystyle\int (x+2)\tan\left(x^2+4x\right)dx=\left(\underbrace{\frac{1}{2}}_{correction}\right)\left(-\ln\left|x^2+4x\right|\right)+C=-\frac{1}{2}\ln\left|x^2+4x\right|+C$

Remember, if there is more than one trig function, hopefully one is the derivative of the other, in which case the trig function is just the passive "inside" guy, and the solution involves the power rule, NOT the actual trig function. Here's an example.

$$\int \sec^2 2x\tan 2x dx=\int\left(\sec 2x\right)\left(\sec 2x\tan 2x\right)dx=\left(\underbrace{\frac{1}{2}}_{correction}\right)\left(\underbrace{\frac{1}{2}}_{power\ rule}\right)\left(\sec 2x\right)^2+C=\frac{1}{4}\sec^2 2x+C$$

This one could have been done differently:

$$\int \sec^2 2x\tan 2x dx=\int\left(\sec^2 2x\right)\left(\tan 2x\right)^1dx=\left(\underbrace{\frac{1}{2}}_{correction}\right)\left(\underbrace{\frac{1}{2}}_{power\ rule}\right)\left(\tan 2x\right)^2+C=\frac{1}{4}\tan^2 2x+C$$

(verify that these are equivalent by using the Pythagorean Identity: $\tan^2 u + 1 = \sec^2 u$)

Let's look at the last independent rule: exponentials!

Remember, if the exponential is in the integrand only once as a single factor, then the guess will likely be itself. If it is present more than once, then it is probably the "inside" function, and the integral will involve the "outside" function.

Ex) $\int x^2 e^{3x^3}\,dx = \left(\underbrace{\frac{1}{9}}_{correction}\right)\left(e^{3x^3}\right) + C$

All the rules and types of functions can now coexist happily in any integral.

Ex) $\int \left(\sin^2 x\right)\left(e^{3\cos^3 x}\right)dx = \left(\underbrace{-\frac{1}{9}}_{correction}\right)\left(e^{3\cos^3 x}\right) + C$

Remember that if the base is not *e*, whether it is log or exponential, you just have to contend with either multiplying or dividing appropriately by the natural log of the base.

Ex) $\int \left(e^x\right)\left(2^{e^x}\right)dx = \left(\underbrace{\frac{1}{\ln 2}}_{correction}\right)\left(2^{e^x}\right) + C$

(notice that *e* is in there twice, so the integral involved the "outside" function "two to the 'somethingth' power")

Now, what do you try when the derivative of the "inside" function is off by more than just a scalar multiple? Try formal *u*-substitution.

Third: Try formal *u*-substitution (change of variables) when *du* is off by a factor of *x* and it is easy to solve for *x* in terms of *u*.

Ex) $\int x\sqrt{2x-1}dx$. Let $u = 2x - 1 \Rightarrow du = 2dx \Rightarrow dx = \frac{1}{2}du$, and $x = \frac{u+1}{2}$.

Substituting gives us $\int \left(\frac{u+1}{2}\right)\left(\sqrt{u}\right)\left(\frac{1}{2}du\right) = \frac{1}{4}\int (u+1)\left(u^{1/2}\right)du = \frac{1}{4}\int u^{3/2} + u^{1/2}du = \ldots$

Finishing in terms of *u* then plugging back in terms of *x* yields the answer.
Notice because *u* was linear, it was easy to solve for *x*. This is an ideal situation to use *u*-substitution. Once you get it in terms of *u*, all of the above rules apply. You basically start over, but hopefully with something a little more recognizable. Look to do a lot of algebra.

Ex) $\int \frac{x^2 - 1}{\sqrt{2x-1}}dx$. Let $u = 2x - 1 \Rightarrow du = 2dx \Rightarrow dx = \frac{1}{2}du$, and $x = \frac{u+1}{2}$.

Substituting gives us

$$\int \frac{\left(\frac{u+1}{2}\right)^2 - 1}{\sqrt{u}}\left(\frac{1}{2}du\right) = \frac{1}{2}\int \frac{\frac{u^2+2u+1}{4} - 1}{u^{1/2}}du = \frac{1}{2}\int \frac{\frac{u^2+2u+1-4}{4}}{u^{1/2}}du = \frac{1}{8}\int u^{3/2} + 2u^{1/2} - 3u^{-1/2}du = \ldots$$

Now the method of last resort: Inverse Trig.

Fourth: If all else fails, or you spot the pattern from the get-go, use the Inverse Trig rules.

Ex) $\displaystyle \int \frac{e^x}{3+e^{2x}}dx = \int \frac{e^x}{\left(\sqrt{3}\right)^2 + \left(e^x\right)^2}dx = \frac{1}{\sqrt{3}}\arctan\frac{e^x}{\sqrt{3}} + C$

Ex) $\displaystyle \int \frac{1}{x\sqrt{2-\ln^2 2x}}dx = \int \frac{\frac{1}{x}}{\sqrt{\left(\sqrt{2}\right)^2 - \left(\ln 2x\right)^2}}dx = \arcsin\frac{\sqrt{2}}{\ln 2x} + C$

(remember: $\displaystyle \frac{d}{dx}\ln 2x = \frac{1}{x}$)

Ex) $\displaystyle \int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{1}{\sqrt{\left(e^x\right)^2 - 1^2}}dx$

This cannot be an Arcsin because the function of x is first under the radical. It looks more like an Arcsec, but it is missing the piece in front of the radical. Since you are off by more than a scalar multiple, try formal u substitution.

Let $\displaystyle u = e^x \Rightarrow du = e^x dx \Rightarrow dx = \frac{1}{e^x}du \Rightarrow dx = \frac{1}{u}du$

Substituting, we get

$\displaystyle \int \frac{1}{\sqrt{u^2-1^2}}\left(\frac{1}{u}du\right) = \int \frac{1}{u\sqrt{u^2-1^2}}du = \frac{1}{1}arc\sec\frac{|u|}{1} + C = arc\sec e^x + C$, since $e^x > 0, \forall x \in \Re$.

Now that we've seen all the rules, lets review some common tricks that are not in themselves methods, but are clever tricks that can be employed along the way to help make the integrand fit a rule or a multitude of rules.

Trick 1: Expanding

Ex) $\displaystyle \int \left(x^4 - x\right)\left(x^4+1\right)dx = \int \left(x^8 + x^4 - x^5 - x\right)dx = \frac{1}{9}x^9 + \frac{1}{5}x^5 - \frac{1}{6}x^6 - \frac{1}{2}x^2 + C$

Trick 2: Split a single fraction up into two

Ex)

$$\int \frac{x+2}{\sqrt{4-x^2}}dx = \int \frac{x}{\sqrt{4-x^2}} + \frac{2}{\sqrt{4-x^2}}dx = \frac{1}{2}(2)\left(4-x^2\right)^{1/2} + 2\arcsin\frac{x}{2} + C = \sqrt{4-x^2} + 2\arcsin\frac{x}{2} + C$$

Trick 3: Long division (in a rational function when the degree of the numerator is ≥ the degree of the denominator.) This can be combined in conjunction with other methods.

Ex)

$$\int \frac{3x^3-2}{x^2+4}dx \Rightarrow x^2+4\,\overset{\displaystyle 3x}{\overline{\left)3x^3-2\right.}} \Rightarrow \int 3x - \frac{12x+2}{x^2+4}dx = 3\int x dx - 12\int \frac{x}{x^2+4}dx - 2\int \frac{1}{x^2+4}dx$$

$$\begin{array}{l} -3x^3-12x \\ \hline -12x-2 \end{array}$$

$$= \frac{3}{2}x^2 - \frac{12}{2}\ln\left(x^2+4\right) - \frac{2}{2}\arctan\frac{x}{2} + C = \frac{3}{2}x^2 - 6\ln\left(x^2+4\right) - \arctan\frac{x}{2} + C.$$

Trick 4: Completing the square

Ex)

$$\int \frac{dx}{\sqrt{-16x^2+16x-3}} = \int \frac{dx1}{\sqrt{-16\left(x^2 - x + \underline{\left(\frac{-1}{2}\right)^2} - \underline{\left(\frac{-1}{2}\right)^2}\right) - 3}}$$

$$= \int \frac{dx}{\sqrt{-16\left(x^2 - x + 1/4\right) + \underset{undistribute}{16(1/4)} - 3}}dx = \int \frac{dx}{\sqrt{-16(x-1/2)^2+1}} = \int \frac{dx}{\sqrt{1-16(x-1/2)^2}}$$

$a = 1, u = 4(x-1/2) = 4x-2, du = 4dx$

$$= \left(\underset{correction}{\frac{1}{4}}\right)\left(\underset{rule}{\frac{1}{1}}\right)\arctan\left(\frac{4x-2}{1}\right) + C = \frac{1}{4}\arctan(4x-2) + C$$

Trick 5: Adding a clever form of zero

Ex) $\int \frac{2x}{x^2+2x+1}dx$.

Long division is not an option. The derivative of the bottom would be exactly the factor in top IF we added TWO. We can do this, so long as we immediately subtract it too.

$$= \int \frac{2x+2-2}{x^2+2x+1}dx = \int \frac{2x+2}{x^2+2x+1} - \frac{2}{x^2+2x+1}dx = \int \frac{2x+2}{x^2+2x+1}dx - 2\int \frac{1}{(x+1)^2}dx$$

$$= \ln(x^2+2x+1) - 2\frac{(x+1)^{-1}}{-1} + C$$
$$= \ln(x^2+2x+1) + \frac{2}{x+1} + C$$

Trick 6: Using Trigonometric Identities

Ex) $\int \cot^2 x dx = \int \csc^2 x - 1 dx = -\cot x - x + C$

Trick 7: Multiplying by a clever form of one (conjugate)

$$\int \frac{1}{1+\sin x}dx = \int \frac{1}{1+\sin x}\left(\frac{1-\sin x}{1-\sin x}\right)dx = \int \frac{1-\sin x}{1-\sin^2 x}dx = \int \frac{1-\sin x}{\cos^2 x}dx = \int \sec^2 x - \frac{\sin x}{\cos^2 x}dx$$

Ex) $= \int \sec^2 x - \left(\frac{\sin x}{\cos x}\right)\left(\frac{1}{\cos x}\right)dx = \int \sec^2 x - \sec x \tan x dx$

$$= \tan x - \sec x + C$$

These are all the techniques you are required to know and be proficient at in my class. There are many other techniques you might encounter as you continue through your mathematics education, including Integration by Parts, Trigonometric Substitution, Partial Fractions, and by Tables (or next to windows)

Remember, all of these Indefinite Integral can be easily be made into definite integrals. The only difference would be the need to evaluate the problem at the end. Never forget, though, the geometric meaning of the definite integral: it is the area under the curve over the specified interval.

Happy Integrating!!