Euler's Method

1707 - 1783

Up to this point practically every differential equation that we've been presented with could be solved. The problem with this is that these are the exceptions rather than the rule. The vast majority of first order differential equations can't be solved.

In order to teach you something about solving first order differential equations we've had to restrict ourselves down to the fairly restrictive cases of linear, separable, and/or exact differential equations. Most first order differential equations however fall into none of these categories. In fact even those that are separable or exact cannot always be solved for an explicit solution. Without explicit solutions to these it would be hard to get any information about the solution.

So what do we do when faced with a differential equation that we can't solve? The answer depends on what you are looking for. If you are only looking for long-term behavior of a solution you can always sketch a slope (direction) field. This can be done without too much difficulty for some fairly complex differential equations that we can't solve to get exact solutions.

The problem with this approach is that it's only really good for getting general *trends* in solutions and for long-term behavior of solutions. There are times when we will need something more. Maybe we need to determine how a specific solution behaves, including some values that the solution will take.

In these cases we resort to numerical methods that will allow us to approximate solutions to differential equations. There are many different methods that can be used to approximate solutions to a differential equation. We are going to look at one of the oldest and easiest to use here. This method was originally devised by Euler and is called, oddly enough, Euler's Method.

Note: When Euler's method is not sufficiently accurate, a number of other techniques, including popular methods such as the Runge-Kutta or Adams methods, may also be used. For our purposes, we will only consider Euler's method.

Euler's method basically involves "walking out along a tightrope" form an initial point along it's tangent line. Instead of walking along the same line, however, we change tangent lines with each step (of length Δx). This involves recalculating the point and slope after each step. This will produce a much more accurate approximation that simply using the original tangent line.

The process it self is pretty easy and repetitive, and it is easier demonstrated with an example rather than a complicated formula.

Here's the "machinery" you will need to make it work:

- We first must designate the number of equal steps we would like to take. Call this number *n*.

- Next, if a is our intitial *x*-value, and b is our desired *x*-value to find our approximation at, we calculate Δx the conventional way:

$$\Delta x = \frac{b-a}{n}$$

- Recall slope: $m = \frac{\Delta y}{\Delta x}$, solving for Δy, we get

$$\Delta y = m(\Delta x)$$

- We can now proceed. The following chart will make things easier.

 (note: the first *x* and *y* used are the initial condition. $\frac{dy}{dx}$ will be given)

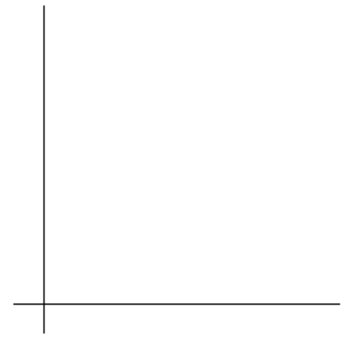
x	y	$m = \left.\frac{dy}{dx}\right\vert_{(x,y)}$	$\Delta y = m(\Delta x)$	$y_{new} = y + \Delta y$

The next *x*-value used will be the previous one PLUS Δx, and the next *y*-value will be the y_{new} from the previous row. You continue this process until you are staring at your desired value of *x* and *y* in the first two columns.

Let's give it a try . . .

Ex. Given the differential equation $\frac{dy}{dx} = x - 2$ and $y(0) = 5$.

(A) Find an approximation for $y(0.8)$ by using Euler's method with two equal steps. Sketch your solution.



(B) Solve the differential equation $\frac{dy}{dx} = x - 2$ with the initial condition $y(0) = 5$, and use your solution to find $y(0.8)$.

Ex. Assume that f and f' have the values given in the table. Use Euler's method with two equal steps to approximate the value of $f(2.6)$.

x	3	2.8	2.6
$f'(x)$	0.4	0.7	0.9
$f(x)$	2		

Worksheet on Euler's Method

Work the following on notebook paper, showing all steps.

1. Answer the following questions.
   (A) Given the differential equation $\frac{dy}{dx} = x + 2$ and $y(0) = 3$. Find an approximation for $y(1)$ by using Euler's method with two equal steps. Sketch you solution.
   (B) Solve the differential equation $\frac{dy}{dx} = x + 2$ with the initial condition $y(0) = 3$, and use your solution to find $y(1)$.
   (C) The error in using Euler's Method is the difference between the approximate value and the exact value. What was the error in your answer? How could you produce a smaller error using Euler's Method?

2. Suppose a continuous function f and its derivative f' have values that are given in the following table. Given that $f(2) = 5$, use Euler's Method with two steps of size $\Delta x = 0.5$ to approximate the value of $f(3)$.

x	2.0	2.5	3.0
$f'(x)$	0.4	0.6	0.8
$f(x)$	5		

3. Given the differential equation $\frac{dy}{dx} = \frac{1}{x+2}$ and $y(0) = 1$, find an approximation of $y(1)$ using Euler's Method with two steps and step size $\Delta x = 0.5$.

4. Given the differential equation $\frac{dy}{dx} = x + y$ and $y(1) = 3$, find an approximation of $y(2)$ using Euler's Method with two equal steps.

5. The curve passing through $(2,0)$ satisfies the differential equation $\frac{dy}{dx} = 4x + y$. Find an approximation to $y(3)$ using Euler's Method with two equal steps.

6. Assume that f and f' have the values given in the table. Use Euler's Method with two equal steps to approximate the value of $f(4.4)$.

x	4	4.2	4.4
$f'(x)$	-0.5	-0.3	-0.1
$f(x)$	2		

7. The table gives selected values for the derivative of a function f on the interval $-2 \le x \le 2$. If $f(-2) = 3$ and Euler's Method with a step size of 1.5 is used to approximate $f(1)$, what is the resulting approximation?

x	$f'(x)$
-2	-0.8
-1.5	-0.5
-1	-0.2
-0.5	0.4
0	0.9
0.5	1.6
1	2.2
1.5	3
2	3.7

8. Let $y = f(x)$ be the particular solution to the differential equation $\frac{dy}{dx} x + 2y$ with the initial condition $f(0) = 1$. Use Euler's Method, starting at $x = 0$ with two steps of equal size to approximated $f(-0.6)$.

Answers

1. (A) 5.25
   (B) 5.5
   (C) Error = 0.25, use smaller steps
2. 5.5
3. 1.45
4. 8.25
5. 11
6. 1.84
7. 2.4
8. 0.25

Korpi

Sample AP Calculus Free-Response questions

(NO CALCULATOR)
1998 AP Calculus AB Free-Response Question

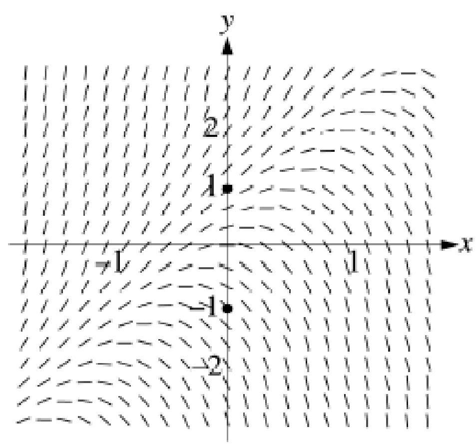
4. Let f be a function with $f(1) = 4$ such that for all points (x, y) on the graph of f the slope is given by $\frac{3x^2+1}{2y}$.

(a) Find the slope of the graph of f at the point where $x = 1$.
(b) Write an equation of the line tangent to the graph of f at $x = 1$ and use it to approximate $f(1.2)$.
(c) Find $f(x)$ by solving the separable differential equation $\frac{dy}{dx} = \frac{3x^2+1}{2y}$ with the initial condition $f(1) = 4$.
(d) Use your solution from part (c) to find $f(1.2)$

2002 AP Calculus BC-5 Free-Response Question
(NO CALCULATOR)

Consider the differential equation $\frac{dy}{dx} = 2y - 4x$.

(a) The slope field for the given differential equation is provided. Sketch the solution curve that passes through the point $(0, 1)$ and sketch the solution curve that passes through the point $(0, -1)$.
(Note: Use the slope field provided in the pink test booklet.)



(b) Let f be the function that satisfies the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

(c) Find the value of b for which $y = 2x + b$ is a solution to the given differential equation. Justify your answer.

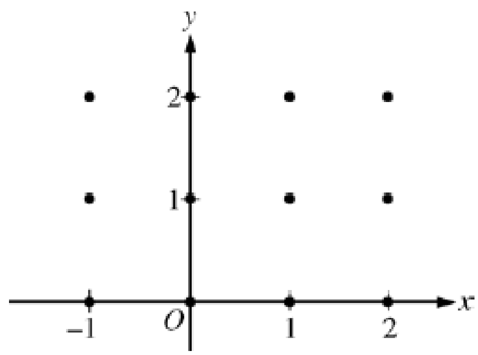
(d) Let g be the function that satisfies the given differential equation with the initial condition $g(0) = 0$. Does the graph of g have a local extremum at the point $(0, 0)$? If so, is the point a local maximum or a local minimum? Justify your answer.

2005 AP Calculus BC-4 Free-Response Question
(NO CALCULATOR)

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and sketch the solution curve that passes through the point $(0, 1)$.

(Note: Use the axes provided in the pink test booklet.)



(b) The solution curve that passes through the point $(0, 1)$ has a local minimum at $x = \ln\left(\frac{3}{2}\right)$. What is the y-coordinate of this local minimum?

(c) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(-0.4)$. Show the work that leads to your answer.

(d) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine whether the approximation found in part (c) is less than or greater than $f(-0.4)$. Explain your reasoning.