Logistic Growth

Recall that things that grew exponentially had a rate of change that was proportional to the value itself. This translated into the following differential equation and solution:

$$\frac{dy}{dt} = ky \rightarrow y = Ce^{kt}$$

An example of this type of growth would be money growing at a specific rate, *k* for amount of time *t*. The more money (*y*) you have, the faster you earn it ($\frac{dy}{dt}$).

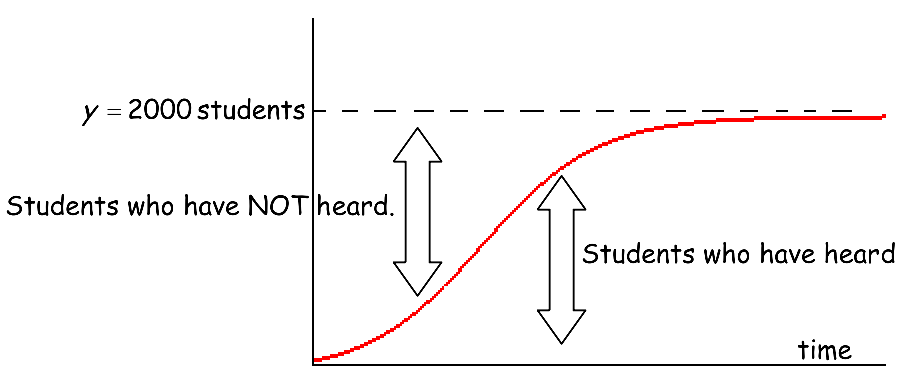
Some things that start out growing (or decaying) exponentially, however, do not continue to do so indefinitely. Instead, they eventually taper off toward some limiting value. For example, hot coffee cools rapidly at first, but as it approaches room temperature, it cools more gradually *approaching* the temperature of the room. Or . . .

Populations with unlimited resources can grow exponentially indefinitely. However, in most real populations, both food and disease become important as conditions become crowded. There is an *upper limit* to the number of individuals the environment can support. Ecologists refer to this as the ***"carrying capacity"*** of the environment.

This typed of growth/decay follows a modified exponential model called Logistic Growth.

In this case, the rate at which things grow is JOINTLY proportional to the amount at any given time AND the difference of that amount from the "carrying capacity" value.

To illustrate this, imagine the scenario where you just heard a rumor at your school: "Math is fun!!" You want to spread that rumor to ALL 2000 of the students at your school. You know how rumors spread . . . . like wildfire, or more correctly, logistically. The rumor spreads slowly, but soon there are more who know, so there are more people to help continue spreading the rumor (this is strictly exponential growth). As more and more students find out, however, their will be fewer and fewer students who HAVEN'T heard, so the rumor CANNOT continue to spread at the same rate. The rumor will continue to spread at a DECREASING rate until all 2000 students know. The graph of such a scenario might look something like this:



The resulting curve resembles an "S" curve. It has a Horizontal Asymptote at $y = L$, where L is the carrying capacity, in this example, 2000. There is another Horizontal Asymptote on the x-axis, $y = 0$ from the exponential contribution. This translates mathematically as:

$$\lim_{t \to \infty} y = L$$

Notice that the graph is monotonic increasing, but changes from increasing at an INCREASING rate to increasing at a DECREASING rate about halfway between the Horizontal Asymptotes. That is because it does!!!

For a standard logistic curve, the inflection point is at $y = \frac{L}{2}$

Let's see if we can't set up and solve a differential equation based on what we know: The rate of change of y (*the rumor spreads at a rate that*) is jointly proportional to y (*students who know*) and $L - y$ (*students who do NOT know*).

$$\frac{dy}{dt} = ky(L - y)$$

$$\int \frac{1}{y(L-y)}\,dy = \int k\,dt \quad \text{(separate variables)}$$

$$\int \frac{1/L}{y} + \frac{1/L}{L-y}\,dy = \int k\,dt \qquad \text{(partial fractions)}$$

$$\frac{1}{L}\left(\ln|y| - \ln|L-y|\right) = kt + C \quad \text{(integrate)}$$

$$\ln\left|\frac{y}{L-y}\right| = Lkt + LC \qquad \text{(let } LC = C\text{)}$$

$$\frac{y}{L-y} = e^{Lkt+C} = \left(e^{Lkt}\right)\left(e^{C}\right) = Ce^{Lkt}$$

$$y = Ce^{Lkt}(L - y)$$

$$y = LCe^{Lkt} - yCe^{Lkt}$$

$$y\left(1 + Ce^{Lkt}\right) = LCe^{Lkt}$$

$$y = \frac{LCe^{Lkt}}{1 + Ce^{Lkt}}\left(\frac{\left(Ce^{Lkt}\right)^{-1}}{\left(Ce^{Lkt}\right)^{-1}}\right)$$

$$y = \frac{L}{1 + \left(C^{-1}\right)e^{-Lkt}} \qquad \text{(let } C^{-1} = C\text{)}$$

$$y = \frac{L}{1 + Ce^{-Lkt}}$$

It is important to note that the value of C, which was the y-intercept for exponential growth is not necessarily the y-intercept for logistic!!!

So now we have the following differential equation and its solution:

For logistic growth: ***(Memorize this)***

$$\frac{dy}{dt} = ky\left(L-y\right) \rightarrow y = \frac{L}{1+Ce^{-Lkt}}$$

******Notice the prominence of our carrying capacity value L in each form of the equation. This is very important, especially when asked for the **limit at infinity** OR when asked to find the y-value when the y-values are **increasing most rapidly** (i.e. the inflection value: $y = \frac{L}{2}$).*

Let's do some examples.

Ex. 1:

The population of Alaska since from 1900 to 2000 can be modeled by the following logistic equation.

$$P(t) = \frac{895598}{1+71.57e^{-0.0516t}}$$

where P is the population and t years after 1900, with $t = 0$ corresponding to 1900.

(A) What is the predicted population of Alaska in 2010?
Answer: 2010 corresponds to $t = 120$. From the calculator,

$$P(120) \approx 781{,}253 \text{ Alaskans}$$

(B) How fast was the population of Alaska changing in 1920? In 1960? In 1999?
Answer: We want to know the value of the derivative at $t = 20, 60$, and 99.
From the calculator, with the equation in the calculator as $Y1$, we can arrive at:

- $P'(20) = nderiv(Y1, x, 20) \approx 1678$ people per year
- $P'(40) = nderiv(Y1, x, 40) \approx 4123$ people per year
- $P'(99) = nderiv(Y1, x, 99) \approx 9742$ people per year

(C) When was Alaska growing the fastest, and what was the population then?
Answer: This occurs at the inflection point, or when $y = L/2$. Since the equation given is written in standard logistic form, $L = 895598$.
The population at this time is therefore

$$P = \frac{895{,}598}{2} = 447{,}799 \text{ Alaskans}$$

Setting this value equal to the equation and solving for t on the calculator, we get:

$$t \approx 83 \rightarrow 1983$$

So, around 1983, the population of Alaska was growing at fastest rate ($P'(83) \approx 11{,}553$ people per year), and its population at that time was approximately 447,799.

(D) What information does the equation tell us about the population of Alaska in the long run?
Answer: for this we look way, way, way down the road, looking to the end of time, as $t \rightarrow \infty$.

$$\lim_{t \to \infty} P(t) = L = 895{,}598$$

Based on the model, the population of Alaska should stabilize at about 895,600 people in the long run, although we cannot be this certain since human population is dependent on too many variables that can, and will, change over time.

The AP exam loves to ask questions that require you to recognize the parameters of logistic growth for either the equation or the differential equation written in a DIFFERERNT FORMAT. This requires you to manipulate the equation to fit one of the two standard forms below:

$$\frac{dy}{dt} = ky\left(L - y\right) \rightarrow y = \frac{L}{1 + Ce^{-Lkt}}$$

Ex. 2:
Suppose that a population develops according to the logistic differential equation $\frac{dp}{dt} = 0.2P - 0.002P^2$, where t is measured in weeks, $t \geq 0$.

(A) If $P(0) = 5$, what is $\lim_{t \to \infty} P(t)$?

(B) If $P(0) = 60$, what is $\lim_{t \to \infty} P(t)$?

(C) If $P(0) = 120$, what is $\lim_{t \to \infty} P(t)$?

(D) Sketch the solution curves for (A), (B), and (C). Which one has an inflection point?

Ex. 3:

The rate at which the flu spreads through a community is modeled by the logistic differential equation $\frac{dp}{dt} = 0.001P(3000 - P)$, where t is measured in days, $t \geq 0$.

(A) If $P(0) = 50$, solve for P as a function of t.

(B) Use your solution to (A) to find the size of the population when $t = 2$ days.

(C) Use your solution to (A) to find the number of days that have occurred when the flu is spreading the fastest.

WORKSHEET FOR LOGISTIC GROWTH

Work the following on notebook paper. Use your calculator on 4(B), 4(C), and 5(C) only.

1. Suppose the population of bears in a national park grows according to the logistic differential equation $\frac{dP}{dt} = 5P - 0.002P^2$, where P is the number of bears at time t in years.
   (A) If $P(0) = 100$, then $\lim_{t \to \infty} P(t) =$ ____________. Sketch the graph of $P(t)$. For what values of P is the graph of P increasing? decreasing? Justify your answer.
   (B) If $P(0) = 1500$, $\lim_{t \to \infty} P(t) =$ ____________. Sketch the graph of $P(t)$. For what values of P is the graph of P increasing? decreasing? Justify your answer.
   (C) If $P(0) = 3000$, $\lim_{t \to \infty} P(t) =$ ____________. Sketch the graph of $P(t)$. For what values of P is the graph of P increasing? decreasing? Justify your answer.
   (D) How many bears are in the park when the population of bears is growing the fastest? Justify your answer.

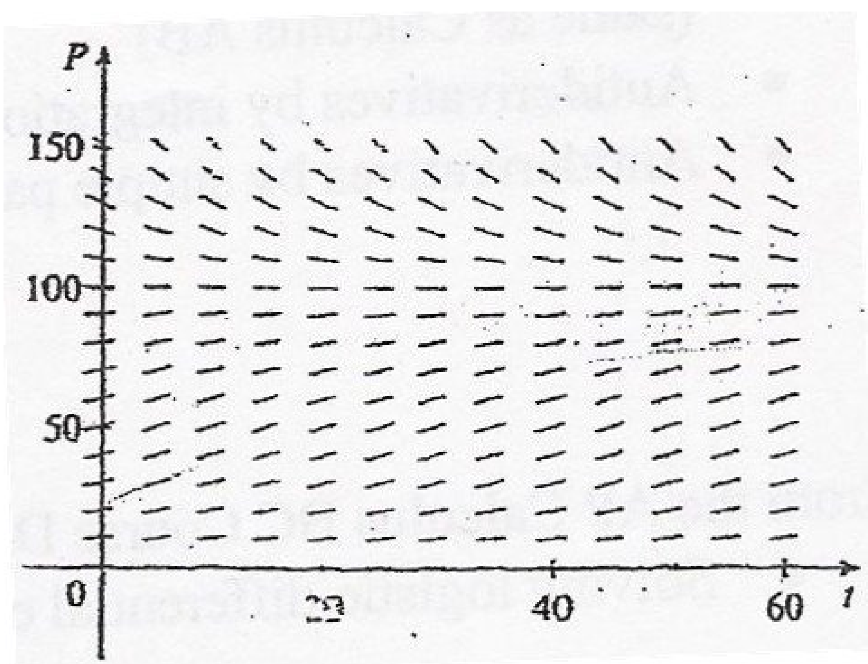
2. (1998 BC Multiple Choice)
   The population $P(t)$ of a species satisfies the logistic differential equation $\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$, where the initial population is $P(0) = 3000$ and t is the time in years.
   What is $\lim_{t \to \infty} P(t)$?
   (A) 2500 (B) 3000 (C) 4200 (D) 5000 (E) 10,000

3. Suppose a population of wolves grows according to the logistic differential equation $\frac{dP}{dt} = 3P - 0.01P^2$, where P is the number of wolves at time t, in years. Which of the following statements are true?
   I. $\lim_{t \to \infty} P(t) = 300$
   II. The growth rate of the wolf population is greatest at $P = 150$.
   III. If $P > 300$, the population of wolves is increasing.
   (A) I only (B) II only (C) I and II only (D) II and III only (E) I, II, and III

4. A population of animals is modeled by a function P that satisfies the logistic differential equation $\frac{dP}{dt} = 0.01P(100 - P)$, where t is measured in years.
   (A) If $P(0) = 20$, solve for P as a function of t.
   (B) Use your answer to (A) to find P when $t = 3$ years.
   (C) Use your answer to (A) to find t when $P = 80$ animals.

5. The rate at which a rumor spreads through a high school of 2000 students can be modeled by the differential equation $\frac{dP}{dt} = 0.003P(2000 - P)$, where P is the number of students who have heard the rumor t hours after 9AM.
    (A) How many students have heard the rumor when it is spreading the fastest?
    (B) If $P(0) = 5$, solve for P as a function of t.
    (C) Use your answer to (B) to determine how many hours have passed when the rumor is spreading the fastest.
    (D) Use your answer to (B) to determine the number of people who have heard the rumor after two hours.

6. Suppose that a population develops according to the logistic equation $\frac{dP}{dt} = 0.05P - 0.0005P^2$ where t is measured in weeks.
    (A) What is the carrying capacity?
    (B) A slope field for this equation is shown below.
        I. Where are the slopes close to zero?
        II. Where are they largest?
        III. Which solutions are increasing?
        IV. Which solutions are decreasing?
    (C) I. Use the slope field to sketch solutions for initial populations of 20, 60, and 120.
        II. What do these solutions have in common?
        III. How do they differ?
        IV. Which solutions have inflection points?
        V. At what population level do they occur?



7. (A) On the slope field shown below, for $\frac{dP}{dt} = 3P - 3P^2$, sketch three solution curves showing different types of behavior for the population *P*.

(B) Describe the meaning of the shape of the solution curves for the population.

I. Where is *P* increasing?

II. Where is *P* decreasing?

III. What happens in the long run?

IV. Are there any inflection points? If so, where?

V. What do the inflection points mean for the population?