SERIES DAY 10
LAGRANGE ERROR BOUND

**Lagrange Form of the Remainder (also called Lagrange Error Bound or Taylor's Theorem Remainder)**
When a Taylor polynomial is used to approximate a function, we need a way to see how accurately the polynomial approximates the function.

$f(x) = P_n(x) + R_n(x)$ so $R_n(x) = f(x) - P_n(x)$

Written in words:
Function = Polynomial Approximation + Remainder, so Remainder = Function – Polynomial Approximation

**Taylor's Theorem**: If a function *f* is differentiable through order $n+1$ in an interval containing *c*, then for each *x* in the interval, there exists a number *z* between *x* and *c* such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

where the remainder $R_n(x)$ (or error) is given by $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$, the **Lagrange Remainder**

Historically, the remainder was not due to Taylor but to a French mathematician, Joseph Louis Lagrange (1736-1813). For this reason, $R_n(x)$ is called the **Lagrange form** of the remainder.

When applying Taylor's Formula, we would not expect to be able to find the exact value of $z$. Rather, we would attempt to find bounds for the derivative $f^{(n+1)}(z)$ (the *y*-value) from which we will be able to tell how large the remainder $R_n(x)$ is.

Ex1. Let *f* be a function with 5 derivatives on the interval $[2,3]$. Assume that $\left|f^{(5)}(x)\right| < 0.2$ for all *x* in the interval $[2,3]$ and that a fourth-degree Taylor polynomial for *f* at $c=2$ is used to estimate $f(3)$

(a) How accurate is this approximation? Give three decimal places.
(b) Suppose that $P_4(3) = 1.763$. Use your answer to (a) to find an interval in which $f(3)$ must lie.
(c) Could $f(3)$ equal 1.778? Why or why not?
(d) Could $f(3)$ equal 1.764? Why or why not?

Ex2. (a) Find the fifth-degree Maclaurin polynomial for $\sin x$. Then use your polynomial to approximated $\sin 1$, and use Taylor's Theorem to find the maximum error for your approximation. Give three decimal places.

(b) Use your answer to (a) to find an interval $[a,b]$ such that $a \le \sin 1 \le b$.

(c) Could $\sin 1$ equal 0.9? Why or why not?

---

Do not use your calculator on Ex. 3

Ex 3. (a) Write the fourth-degree Maclaurin polynomial for $f(x) = e^x$. Then use your polynomial to approximate $e$, and find a Lagrange error bound for the maximum error when $|x| \le 1$.

(b) Use your answer to (a) to find an interval $[a,b]$ such that $a \le e \le b$.

---

Ex4. The function $f$ has derivatives of all orders for all real numbers $x$. Assume that $f(2) = 6$, $f'(2) = 4$, $f''(2) = -7$, $f'''(2) = 8$.

(a) Write the third-degree Taylor polynomial for $f$ about $x = 2$, and use it to approximate $f(2.3)$. Give three decimal places.

(b) The fourth derivative of $f$ satisfies the inequality $\left|f^{(4)}(x)\right| \le 9$ for all $x$ in the closed interval $[2, 2.3]$. Use the Lagrange error bound on the approximation of $f(2.3)$ found in part (a) to find an interval $[a,b]$ such that $a \le f(2.3) \le b$. Give three decimal places.

(c) Based on the information above, could $f(2.3)$ equal 6.992? Explain why or why not.