SERIES DAY 4

Comparison of Series

Direct Comparison Test

If $a_n \geq 0$ and $b_n \geq 0$,

1) If $\sum_{n=1}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ ______________________________.

2) If $\sum_{n=1}^{\infty} a_n$ diverges and $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} b_n$ ______________________________.

Ex. Determine whether the following converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{n^3+1}$

(b) $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$

(c) $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}-1}$

Limit Comparison Test

If $a_n \geq 0$ and $b_n \geq 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where L is both finite and positive.

Then the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

Ex. Determine whether the following converge or diverge.

(a) $\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$

(b) $\sum_{n=1}^{\infty} \frac{n^4 + 10}{4n^5 - n^3 + 7}$

(c) $\sum_{n=2}^{\infty} \frac{1}{n^3 - 2}$

(d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n - 2}}$

Alternating Series

An alternating series is a series whose terms are alternately positive and negative on consecutive terms.

Examples:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

$$-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\cdots\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n!}$$

In general, just knowing that $\lim_{n\to\infty} a_n = 0$ tells us very little about the convergence of the series $\sum_{n=1}^{\infty} a_n$; however, it turns out that an alternating series must converge if its terms consistently shrink in size and approach zero!!

Alternating Series Test

If $a_n > 0$, then the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converges if both of the following conditions are satisfies:

1) $\lim_{n\to\infty} a_n = 0$

2) $\{a_n\}$ is a decreasing (or Non-increasing) sequence; that is, $a_{n+1} \le a_n$ for all $n > k$, for some $k \in \mathbb{Z}$

Note: This does NOT say that if $\lim_{n\to\infty} a_n \ne 0$ the series DIVERGES by the AST. The AST can ONLY be used to prove convergence. If $\lim_{n\to\infty} a_n \ne 0$, then the series diverges, but by the ***n*th-term** test NOT the AST.

Ex. Determine whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1} n}{2n-1}$

(b) $\sum_{n=1}^{\infty}\frac{(-1)^{n} n}{\ln(2n)}$

(c) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$

(d) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}$

ABOSOLUTE VS CONDITIONAL CONVERGENCE

Theorem:

If the series $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty}a_n$ also converges.

Such a series is called **absolutely convergent**. Notice that if it converges on its "own," the alternator only allows it to converge more "rapidly."

$\sum_{n=1}^{\infty}a_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty}a_n$ converges but $\sum_{n=1}^{\infty}|a_n|$ diverges.

Note: sometimes a mere rearrangement of terms can yield different sums!!!

Ex. Determine whether the given alternating series converges or diverges. If it converges, determine whether it is absolutely convergent or conditionally convergent.

(a) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

(b) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3^n}$

Alternate Series Remainder

Suppose an alternating series satisfies the conditions of the AST, namely that $\lim_{n\to\infty}a_n = 0$ and $\{a_n\}$ is not increasing. If the series has a sum *S*, then $|R_N| = |S - S_n| \le a_{n+1}$, where S_n is the *n*th partial sum of the series.

In other words, if an alternating series satisfies the conditions of the AST, you can approximate the sum of the series by using the *n*th partial sum, S_n, and your error will have an absolute value no greater than the first term left off, a_{n+1}

Ex. Approximate the sum $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}$ by using its first six terms, and find the error. Use your results to find an interval in which *S* must lie.

Ex. Approximate the sum of $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}$ with an error less than 0.001