CALCULUS BC
WORKSHEET ON SERIES AND ERROR
Work the following on notebook paper.

1. Let f be a function that has derivatives of all orders on the interval $(-1, 1)$. Assume $f(0)=1$, $f'(0)=\frac{1}{2}$, $f''(0)=-\frac{1}{4}$, $f'''(0)=\frac{3}{8}$, and $\left|f^{(4)}(x)\right|\le 6$ for all x in the interval (0, 1).
   (a) Find the third-degree Taylor polynomial about $x=0$ for the function f.
   (b) Use your answer to part (a) to estimate the value of $f(0.5)$.
   (c) What is the maximum possible error for the approximation made in part (b)?

2. Let f be the function defined by $f(x)=\sqrt{x}$.
   (a) Find the second-degree Taylor polynomial about $x=4$ for the function f.
   (b) Use your answer to part (a) to estimate the value of $f(4.2)$.
   (c) Find a bound on the error for the approximation in part (b).

3. Let $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{2^n}$ for all x for which the series converges.
   (a) Find the interval of convergence of this series.
   (b) Use the first three terms of this series to approximate $f\left(-\frac{1}{2}\right)$.
   (c) Estimate the error involved in the approximation in part (b). Show your reasoning.

4. Let f be the function given by $f(x)=\cos\left(3x+\frac{\pi}{6}\right)$ and let $P(x)$ be the fourth-degree Taylor polynomial for f about $x=0$.
   (a) Find $P(x)$.
   (b) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{6}\right)-P\left(\frac{1}{6}\right)\right|<\frac{1}{3000}$.

5. Use series to find an estimate for $\int_0^1 e^{-x^2}\,dx$ that is accurate to three decimal places. Justify.

6. Suppose a function f is approximated with a fourth-degree Taylor polynomial about $x=1$. If the maximum value of the fifth derivative between $x=1$ and $x=3$ is 0.01, that is, $\left|f^{(5)}(x)\right|<0.01$, then the maximum error incurred using this approximation to compute $f(3)$ is
   (A) 0.054 (B) 0.0054 (C) 0.26667 (D) 0.02667 (E) 0.00267

7. The maximum error incurred by approximating the sum of the series $1-\frac{1}{2!}+\frac{2}{3!}-\frac{3}{4!}+\frac{4}{5!}-\ldots$ by the sum of the first six terms is
   (A) 0.001190 (B) 0.006944 (C) 0.33333 (D) 0.125000 (E) None of these

8. The Taylor series about $x=5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x=5$ is given by
   $f^{(n)}(5)=\frac{(-1)^n n!}{2^n(n+2)}$ and $f(5)=\frac{1}{2}$. Show that the sixth-degree Taylor polynomial for f about $x=5$ approximates $f(6)$ with an error less than $\frac{1}{1000}$.