1993 and 2003 - Last, 2005 F.R. polar w/only 1 polar M.C. (Skip in Bind)

BC-4

1993

CALCULUS BC, QUESTION NO. 4

Consider the polar curve $r = 2\sin(3\theta)$ for $0 \le \theta \le \pi$.

(a) In the xy-plane provided below, sketch the curve.

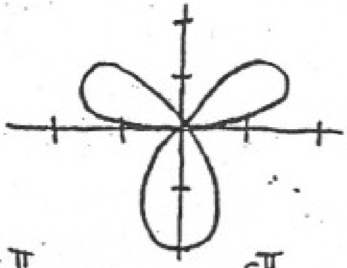
Note: The xy-plane is provided in the pink test booklet only.

(b) Find the area of the region inside the curve.

(c) Find the slope of the curve at the point where $\theta = \frac{\pi}{4}$.

---

a)



2 { 1: 3 leaves; 1: Max $r = 2$ and correct leaf placement }

b) $A = \frac{1}{2}\int_0^{\pi} 4\sin^2 3\theta\, d\theta = \int_0^{\pi} (1-\cos 6\theta)\, d\theta = \theta - \frac{1}{6}\sin 6\theta \Big|_0^{\pi} = \pi$

or $\frac{3}{2}\int_0^{\pi/3} 4\sin^2 3\theta\, d\theta = \ldots = \pi$

or $\frac{6}{2}\int_0^{\pi/6} 4\sin^2 3\theta\, d\theta = \ldots = \pi$

3 { 1: Constant and limits of integration; 1: Integrand; 1: Antidifferentiation and evaluation [must involve $\sin^2(3\theta)$] }

c) $x = 2\sin 3\theta \cos\theta$

$y = 2\sin 3\theta \sin\theta$

$dx/d\theta = -2\sin 3\theta \sin\theta + 6\cos 3\theta \cos\theta$

$dy/d\theta = 2\sin 3\theta \cos\theta + 6\cos 3\theta \sin\theta$

At $\theta = \pi/4$, $dy/d\theta = -2$ and $dx/d\theta = -4$, so

$dy/dx = -2/-4 = 1/2$

OR

$(x^2+y^2)^2 = 6x^2y - 2y^3$

$2(x^2+y^2)(2x + 2y\frac{dy}{dx}) = 6x^2\frac{dy}{dx} + 12xy - 6y^2\frac{dy}{dx}$

At $\theta = \pi/4$, $x = 1$ and $y = 1$ so

$4(2 + 2\,dy/dx) = 6\,dy/dx + 12 - 6\,dy/dx$

$8 + 8\,dy/dx = 12 \Rightarrow dy/dx = 1/2$

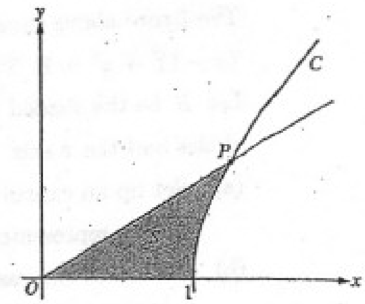
4 {
1: $x = 2\sin 3\theta\cos\theta$, $y = 2\sin 3\theta \sin\theta$
1: $dx/d\theta$
1: $dy/d\theta$
1: Answer

OR

1: Expresses curve in rectangular coordinates
2: Implicit differentiation involving $(x^2+y^2)^p$, $p \ne 0$,
1: Answer using $x=1$ and $y=1$ in student's derivative.
Note: 0/4 if θ not eliminated before diff.
}

142

with calc



The figure above shows the graphs of the line $x = \frac{5}{3}y$ and the curve C given by $x = \sqrt{1+y^2}$. Let S be the shaded region bounded by the two graphs and the x-axis. The line and the curve intersect at point P.

(a) Find the coordinates of point P and the value of $\frac{dx}{dy}$ for curve C at point P.

(b) Set up and evaluate an integral expression with respect to y that gives the area of S.

Split question

(c) Curve C is a part of the curve $x^2 - y^2 = 1$. Show that $x^2 - y^2 = 1$ can be written as the polar equation $r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$.

(d) Use the polar equation given in part (c) to set up an integral expression with respect to the polar angle θ that represents the area of S.

(a) At P, $\frac{5}{3}y = \sqrt{1+y^2}$, so $y = \frac{3}{4}$.

Since $x = \frac{5}{3}y$, $x = \frac{5}{4}$.

$$\frac{dx}{dy} = \frac{y}{\sqrt{1+y^2}} = \frac{y}{x}. \text{ At } P,\ \frac{dx}{dy} = \frac{3/4}{5/4} = \frac{3}{5}.$$

2 : { 1 : coordinates of P; 1 : $\frac{dx}{dy}$ at P }

(b) Area $= \int_0^{3/4} \left(\sqrt{1+y^2} - \frac{5}{3}y\right) dy$

$= 0.346$ or 0.347

3 : { 1 : limits; 1 : integrand; 1 : answer }

(c) $x = r\cos\theta$; $y = r\sin\theta$

$x^2 - y^2 = 1 \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 1$

$$r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$$

2 : { 1 : substitutes $x = r\cos\theta$ and $y = r\sin\theta$ into $x^2 - y^2 = 1$; 1 : isolates r^2 }

(d) Let β be the angle that segment OP makes with the x-axis. Then $\tan\beta = \frac{y}{x} = \frac{3/4}{5/4} = \frac{3}{5}$.

$$\text{Area} = \int_0^{\tan^{-1}(3/5)} \frac{1}{2}r^2\,d\theta$$

$$= \frac{1}{2}\int_0^{\tan^{-1}(3/5)} \frac{1}{\cos^2\theta - \sin^2\theta}\,d\theta$$

2 : { 1 : limits; 1 : integrand and constant }

(but need something for limits. can't leave blank)

NOTE: ** they view it as indef express rather than def. $\int$ → Area. not just "a" to "b" either. must be numbers in domain.

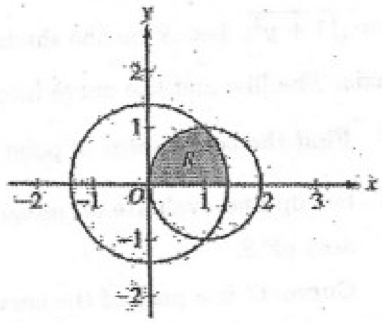
143

AP® CALCULUS BC
2003 SCORING GUIDELINES (Form B)

Question 2

split question

The figure above shows the graphs of the circles $x^2 + y^2 = 2$ and $(x-1)^2 + y^2 = 1$. The graphs intersect at the points $(1,1)$ and $(1,-1)$. Let R be the shaded region in the first quadrant bounded by the two circles and the x-axis.



(a) Set up an expression involving one or more integrals with respect to x that represents the area of R.

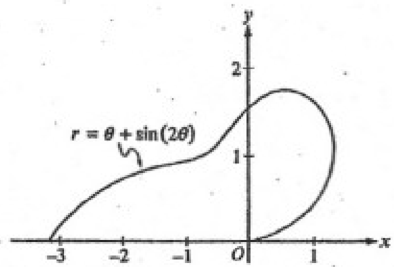
(b) Set up an expression involving one or more integrals with respect to y that represents the area of R.

(c) The polar equations of the circles are $r = \sqrt{2}$ and $r = 2\cos\theta$, respectively. Set up an expression involving one or more integrals with respect to the polar angle θ that represents the area of R.

(a) Area $= \int_0^1 \sqrt{1-(x-1)^2}\,dx + \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx$ OR Area $= \frac{1}{4}(\pi \cdot 1^2) + \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx$	3 : { 1 : integrand for larger circle 1 : integrand or geometric area for smaller circle 1 : limits on integral(s) Note: < −1 > if no addition of terms
(b) Area $= \int_0^1 \left(\sqrt{2-y^2} - \left(1 - \sqrt{1-y^2}\right)\right)dy$	3 : { 1 : limits 2 : integrand < −1 > reversal < −1 > algebra error in solving for x < −1 > add rather than subtract < −2 > other errors
(c) Area $= \int_0^{\pi/4} \frac{1}{2}(\sqrt{2})^2\,d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\,d\theta$ OR Area $= \frac{1}{8}\pi(\sqrt{2})^2 + \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\,d\theta$	3 : { 1 : integrand or geometric area for larger circle 1 : integrand for smaller circle 1 : limits on integral(s) Note: < −1 > if no addition of terms

1244

The curve above is drawn in the xy-plane and is described by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $0 \le \theta \le \pi$, where r is measured in meters and θ is measured in radians. The derivative of r with respect to θ is given by $\frac{dr}{d\theta} = 1 + 2\cos(2\theta)$.



(a) Find the area bounded by the curve and the x-axis.

(b) Find the angle θ that corresponds to the point on the curve with x-coordinate -2.

(c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

(d) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance from the origin. Justify your answer.

(a) Area $= \frac{1}{2}\int_0^{\pi} r^2\, d\theta$

$= \frac{1}{2}\int_0^{\pi} (\theta + \sin(2\theta))^2\, d\theta = 4.382$

$$3: \begin{cases} 1: \text{limits and constant} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$$

(b) $-2 = r\cos(\theta) = (\theta + \sin(2\theta))\cos(\theta)$

$\theta = 2.786$

$$2: \begin{cases} 1: \text{equation} \\ 1: \text{answer} \end{cases}$$

(c) Since $\frac{dr}{d\theta} < 0$ for $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, r is decreasing on this interval. This means the curve is getting closer to the origin.

$$2: \begin{cases} 1: \text{information about } r \\ 1: \text{information about the curve} \end{cases}$$

(d) The only value in $\left[0, \frac{\pi}{2}\right]$ where $\frac{dr}{d\theta} = 0$ is $\theta = \frac{\pi}{3}$.

θ	r
0	0
$\frac{\pi}{3}$	1.913
$\frac{\pi}{2}$	1.571

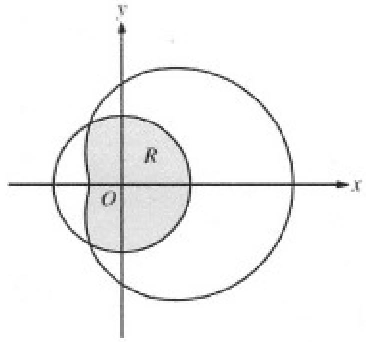
The greatest distance occurs when $\theta = \frac{\pi}{3}$.

$$2: \begin{cases} 1: \theta = \frac{\pi}{3} \text{ or } 1.047 \\ 1: \text{answer with justification} \end{cases}$$

AP® CALCULUS BC
2007 SCORING GUIDELINES

Question 3

The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.



(a) Let R be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of R.

(b) A particle moving with nonzero velocity along the polar curve given by $r = 3 + 2\cos\theta$ has position $(x(t), y(t))$ at time t, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(a) Area $= \frac{2}{3}\pi(2)^2 + \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(3 + 2\cos\theta)^2\,d\theta$ $= 10.370$	4 : { 1 : area of circular sector; 2 : integral for section of limaçon (1 : integrand; 1 : limits and constant); 1 : answer }
(b) $\left.\frac{dr}{dt}\right\rvert_{\theta=\pi/3} = \left.\frac{dr}{d\theta}\right\rvert_{\theta=\pi/3} = -1.732$ The particle is moving closer to the origin, since $\frac{dr}{dt} < 0$ and $r > 0$ when $\theta = \frac{\pi}{3}$.	2 : { 1 : $\left.\frac{dr}{dt}\right\rvert_{\theta=\pi/3}$; 1 : interpretation }
(c) $y = r\sin\theta = (3 + 2\cos\theta)\sin\theta$ $\left.\frac{dy}{dt}\right\rvert_{\theta=\pi/3} = \left.\frac{dy}{d\theta}\right\rvert_{\theta=\pi/3} = 0.5$ The particle is moving away from the x-axis, since $\frac{dy}{dt} > 0$ and $y > 0$ when $\theta = \frac{\pi}{3}$.	3 : { 1 : expression for y in terms of θ; 1 : $\left.\frac{dy}{dt}\right\rvert_{\theta=\pi/3}$; 1: interpretation }