1. C
2. B
3. B
4. B
5. D
6. E
7. A
8. B
9. D
10. B

7

AB Review 07, No Calculator Permitted, unless specified to the contrary.

1. (Calculator Permitted) Let f be the function given by $f(x)=3e^{2x}$ and let g be the function given by $g(x)=6x^3$. At what value of x do the graphs of f and g have parallel tangent lines?

   (A) -0.701 (B) -0.567 (C) -0.391 (D) -0.302 (E) -0.258

2. The radius of a circle is decreasing at a constant rate of 0.1 centimeters per second. In terms of the circumference C, what is the rate of change of the area of the circle, in square centimeters per second?

   (A) $-(0.2)\pi C$ (B) $-(0.1)C$ (C) $-\dfrac{(0.1)C}{2\pi}$ (D) $(0.1)^2 C$ (E) $(0.1)^2 \pi C$

3. (Calculator Permitted) The first derivative of a function f is given by $f'(x)=\dfrac{\cos^2 x}{x}-\dfrac{1}{5}$. How many critical values does f have on the open interval $(0,10)$?

   (A) One (B) Three (C) Four (D) Five (E) Seven

4. $\lim\limits_{x\to\infty}\dfrac{(2x-1)(3-x)}{(x-1)(x+3)}$ is

   (A) -3 (B) -2 (C) 2 (D) 3 (E) nonexistent

5. Let f be the function given by $f(x)=|x|$. Which of the following statements about f are true?

   I. f is continuous at $x=0$.

   II. f is differentiable at $x=0$.

   III. f has an absolute minimum at $x=0$.

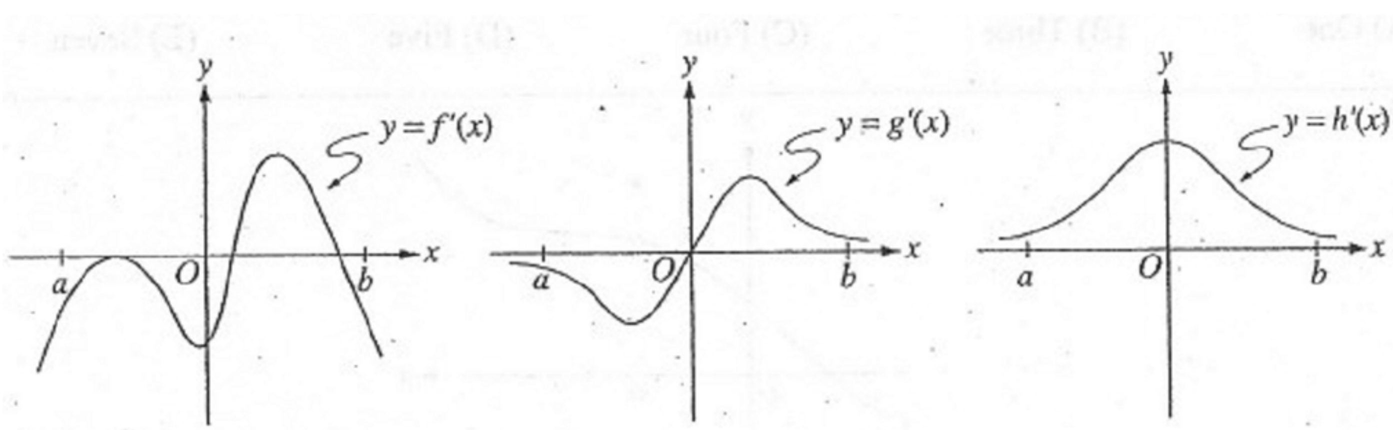
   (A) I only (B) II only (C) III only (D) I and III only (E) II and III only

$\int \cos(2x)\,dx = \frac{1}{2}\sin(2x)+C$

6. If f is a continuous function and if $F'(x)=f(x)$ for all real numbers x, then $\int_1^3 f(2x)\,dx = \frac{1}{2}F(2x)\Big|_1^3$

$\frac{1}{2}[F(6)-F(2)]$

(A) $2F(3)-2F(1)$ (B) $\frac{1}{2}F(3)-\frac{1}{2}F(1)$ (C) $2F(6)-2F(2)$ (D) $F(6)-F(2)$ (E) $\frac{1}{2}F(6)-\frac{1}{2}F(2)$



7. The graphs of the derivatives of the functions f, g, and h are shown above. Which of the functions f, g, or h have a relative maximum on the open interval $a<x<b$?

   (A) f only (B) g only (C) h only (D) f and g only (E) f, g, and h

$e^{\ln x}$

8. If $\frac{dy}{dt} = ky$ and k is a nonzero constant, then y could be

$y = 2e^{kt}$ (A) $\cancel{2e^{kty}}$ (B) $2e^{kt}$ (C) $\cancel{e^{kt}+3}$ (D) $\cancel{kty+5}$ (E) $\cancel{\frac{1}{2}ky^2 + \frac{1}{2}}$

Vert dilation

Horz shift

$e^{(kt+c)} = e^{kt} \cdot e^{c}$

$c=2$

e^{kt+3} ✓

$y = 3 \cdot 4^{kt-12}$ ✓

9. If $f(x) = (x-1)(x^2+2)^3$, then $f'(x) =$

(A) $6x(x^2+2)^2$ (B) $6x(x-1)(x^2+2)^2$ (C) $(x^2+2)^2(x^2+3x-1)$

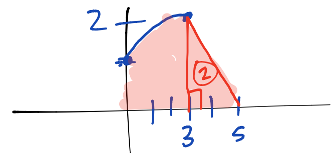
(D) $(x^2+2)^2(7x^2-6x+2)$ (E) $-3(x-1)(x^2+2)^2$

10. A particle moves along the x-axis with velocity given by $v(t) = 3t^2 + 6t$ for time $t \ge 0$. If the particle is at position $x = 2$ at time $t = 0$, what is the position of the particle at $t = 1$?

(A) 4 (B) 6 (C) 9 (D) 11 (E) 12

11. (2003, AB-6) Let f be the function defined by

$$f(x)=\begin{cases}\sqrt{x+1} & \text{for } 0\le x\le 3\\ 5-x & \text{for } 3<x\le 5\end{cases}$$



(a) Is f continuous at $x=3$? Explain why or why not.

$\lim_{x\to 3^-} f(x) = 2 = f(3)$

$\lim_{x\to 3^+} f(x) = 2$

f(x) is continuous @ x=3 since 2 = 2.

$\int_0^3 (x+1)^{1/2}\,dx + \int_3^5 (5-x)\,dx$

$\frac{2}{3}(x+1)^{3/2}\Big|_0^3 + \left(5x-\frac{1}{2}x^2\right)\Big|_3^5$

$\frac{2}{3}\left[4^{3/2}-1^{3/2}\right] + \left(25-\frac{25}{2}\right)-\left(15-\frac{9}{2}\right)$

$\frac{2}{3}[7]$

$\boxed{\frac{14}{3}}$

(b) Find the average value of $f(x)$ on the closed interval $0\le x\le 5$.

$$\text{Avg} = \frac{\int_0^5 f(x)\,dx}{5-0}$$

$$= \boxed{\frac{\frac{14}{3}+2}{5}}$$

$\frac{20}{3}\left(\frac{1}{5}\right)$

$\boxed{\frac{4}{3}}$

(c) Suppose the function g is defined by

$$g(x)=\begin{cases}k\sqrt{x+1} & \text{for } 0\le x\le 3\\ mx+2 & \text{for } 3<x\le 5\end{cases}$$

Where k and m are constants. If g is differentiable at $x=3$, what are the values of k and m?

y-values

$\lim_{x\to 3^-} g(x) = g(3) = 2k$

$\lim_{x\to 3^+} g(x) = 3m+2$

So $\boxed{2k = 3m+2}$

$2(4m) = 3m+2$

$8m-3m=2$

$5m=2$

$\boxed{m=\frac{2}{5}}$

$\boxed{k=\frac{8}{5}}$

$g'(x) = \begin{cases}\frac{k}{2}(x+1)^{-1/2}, & 0\le x\le 3\\ m, & 3<x\le 5\end{cases}$

Slopes

$\lim_{x\to 3^-} g'(x) = g'(3) = \frac{k}{4}$

$\lim_{x\to 3^+} g'(x) = m$

So $\boxed{m=\frac{k}{4}}$

$\boxed{k=4m}$

Question 6

Let f be the function defined by

$$f(x) = \begin{cases} \sqrt{x+1} & \text{for } 0 \le x \le 3 \\ 5 - x & \text{for } 3 < x \le 5. \end{cases}$$

(a) Is f continuous at $x = 3$? Explain why or why not.

(b) Find the average value of $f(x)$ on the closed interval $0 \le x \le 5$.

(c) Suppose the function g is defined by

$$g(x) = \begin{cases} k\sqrt{x+1} & \text{for } 0 \le x \le 3 \\ mx + 2 & \text{for } 3 < x \le 5, \end{cases}$$

where k and m are constants. If g is differentiable at $x = 3$, what are the values of k and m?

(a) f is continuous at $x = 3$ because

$\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^+} f(x) = 2.$

Therefore, $\lim\limits_{x \to 3} f(x) = 2 = f(3).$

$2: \begin{cases} 1: \text{answers "yes" and equates the values of the left- and right-hand limits} \\ 1: \text{explanation involving limits} \end{cases}$

(b) $$\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx$$
$$= \frac{2}{3}(x+1)^{3/2}\Big|_0^3 + \left(5x - \frac{1}{2}x^2\right)\Big|_3^5$$
$$= \left(\frac{16}{3} - \frac{2}{3}\right) + \left(\frac{25}{2} - \frac{21}{2}\right) = \frac{20}{3}$$

Average value: $\frac{1}{5}\int_0^5 f(x)\,dx = \frac{4}{3}$

$4: \begin{cases} 1: k\int_0^3 f(x)\,dx + k\int_3^5 f(x)\,dx \\ \quad (\text{where } k \ne 0) \\ 1: \text{antiderivative of } \sqrt{x+1} \\ 1: \text{antiderivative of } 5 - x \\ 1: \text{evaluation and answer} \end{cases}$

(c) Since g is continuous at $x = 3$, $2k = 3m + 2$.

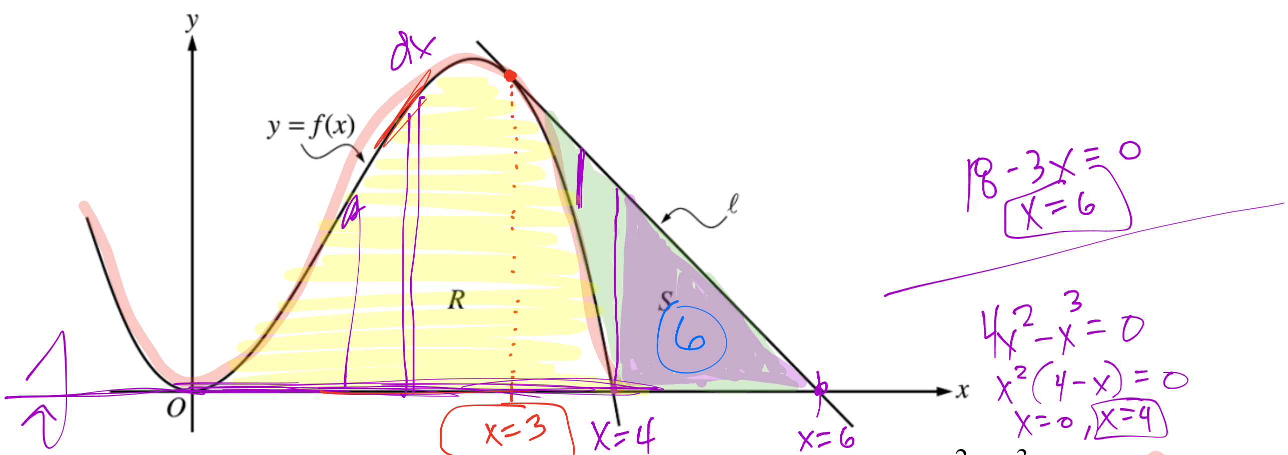
$$g'(x) = \begin{cases} \dfrac{k}{2\sqrt{x+1}} & \text{for } 0 < x < 3 \\ m & \text{for } 3 < x < 5 \end{cases}$$

$\lim\limits_{x \to 3^-} g'(x) = \frac{k}{4}$ and $\lim\limits_{x \to 3^+} g'(x) = m$

Since these two limits exist and g is differentiable at $x = 3$, the two limits are equal. Thus $\frac{k}{4} = m$.

$8m = 3m + 2$; $m = \frac{2}{5}$ and $k = \frac{8}{5}$

$3: \begin{cases} 1: 2k = 3m + 2 \\ 1: \frac{k}{4} = m \\ 1: \text{values for } k \text{ and } m \end{cases}$



12. (2003B, AB/BC-1) (Calculator Permitted) Let f be the function given by $f(x)=4x^2-x^3$, and let ℓ be the line $y=18-3x$, where ℓ is tangent to the graph of f. Let R be the region bounded by the graph of f and the x-axis, and let S be the region bounded by the graph of f, the line ℓ, and the x-axis, as shown above.

(a) Show that ℓ is tangent to the graph of $y=f(x)$ at the point $x=3$.

y-values
$f(3)=36-27=9$
$\ell(3)=18-9=9$
$9=9$ ✓

Slopes
$f'(x)=8x-3x^2$
$f'(3)=24-27=-3$
$\ell'(x)=-3$
$\ell'(3)=-3$
$-3=-3$ ✓

So ℓ is tangent to $f(x)$ at $x=3$.

(b) Find the area of S.

$$\text{Area}=\int_3^4\left((18-3x)-(4x^2-x^3)\right)dx+6$$

$$=7.916$$

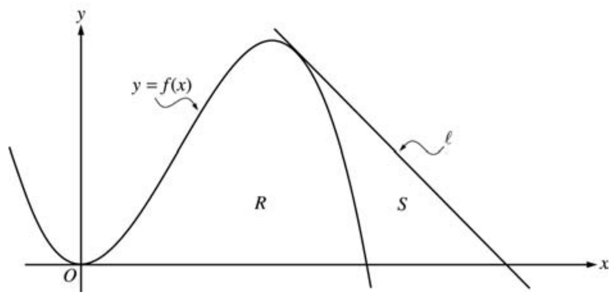
(c) Find the volume of the solid generated when R is revolved about the x-axis.

$$\text{Vol}=\pi\int_0^4\left((4x^2-x^3)-0\right)^2dx$$ ✓

✓ 496.208 ✓

Question 1

Let f be the function given by $f(x) = 4x^2 - x^3$, and let ℓ be the line $y = 18 - 3x$, where ℓ is tangent to the graph of f. Let R be the region bounded by the graph of f and the x-axis, and let S be the region bounded by the graph of f, the line ℓ, and the x-axis, as shown above.



(a) Show that ℓ is tangent to the graph of $y = f(x)$ at the point $x = 3$.

(b) Find the area of S.

(c) Find the volume of the solid generated when R is revolved about the x-axis.

(a) $f'(x) = 8x - 3x^2$; $f'(3) = 24 - 27 = -3$

$f(3) = 36 - 27 = 9$

Tangent line at $x = 3$ is

$y = -3(x - 3) + 9 = -3x + 18,$

which is the equation of line ℓ.

$$2: \begin{cases} 1: \text{finds } f'(3) \text{ and } f(3) \\ 1: \begin{cases} \text{finds equation of tangent line} \\ \text{or} \\ \text{shows } (3,9) \text{ is on both the} \\ \text{graph of } f \text{ and line } \ell \end{cases} \end{cases}$$

(b) $f(x) = 0$ at $x = 4$

The line intersects the x-axis at $x = 6$.

$$\text{Area} = \frac{1}{2}(3)(9) - \int_3^4 \left(4x^2 - x^3\right) dx$$
$$= 7.916 \text{ or } 7.917$$

OR

$$\text{Area} = \int_3^4 \left((18 - 3x) - \left(4x^2 - x^3\right)\right) dx + \frac{1}{2}(2)(18 - 12)$$
$$= 7.916 \text{ or } 7.917$$

$$4: \begin{cases} 2: \text{integral for non-triangular region} \\ \quad 1: \text{limits} \\ \quad 1: \text{integrand} \\ 1: \text{area of triangular region} \\ 1: \text{answer} \end{cases}$$

(c) $$\text{Volume} = \pi \int_0^4 \left(4x^2 - x^3\right)^2 dx$$
$$= 156.038\pi \text{ or } 490.208$$

$$3: \begin{cases} 1: \text{limits and constant} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$$