BC Review FINAL, NO Calculator Permitted (unless stated otherwise)
Do all work on separate notebook paper

1. (a) $\int x\sin(2x)dx = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + C$

By parts

u	dv	+/–
x	sin2x	+
1	$-\frac{1}{2}\cos 2x$	–
0	$-\frac{1}{4}\sin 2x$	+

(b) $\int \frac{dx}{x^2-6x+8} =$

$$\int \frac{1}{(x-4)(x-2)}dx$$

$$\int \left[\frac{1/2}{x-4} - \frac{1/2}{x-2}\right]dx$$

$$\frac{1}{2}\left[\ln|x-4| - \ln|x-2|\right] + C$$

$$\frac{1}{2}\ln\left|\frac{x-4}{x-2}\right| + C$$

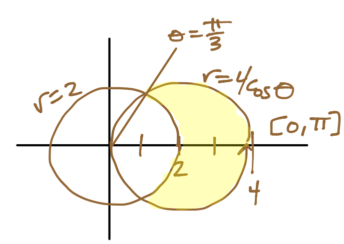
2. Write an integral expression which gives the area of the region inside the polar curve $r = 4\cos\theta$ and outside $r = 2$.

intersect

$4\cos\theta = 2$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$



using symmetry

$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$$

$$\text{Area} = 2\left[\frac{1}{2}\int_0^{\pi/3}\left((4\cos\theta)^2 - (2)^2\right)d\theta\right]$$

$$= \int_0^{\pi/3}\left(16\cos^2\theta - 4\right)d\theta$$

3. Given $\frac{dy}{dx} = \frac{xy}{2}$. Let $f(x)$ be the particular solution to the given differential equation with initial condition $f(0) = 3$. Use Euler's method starting at $x = 0$, with a step size of 0.1, to approximate $f(0.2)$.

$\Delta x = 0.1$

x	y	$m = \frac{dy}{dx}$	$\Delta y = m\Delta x$	$y_{new} = y + \Delta y$
0	3	0	0	3
+0.1 → 0.1	3	$\frac{.3}{2} = .15$	.015	3.015
+0.1 → 0.2	3.015			

So, $f(0.2) \approx 3.015$*

*Thanks, Col. Bird, 5/10/19

4. If $3xy + 2x = 1 - 2y$, then when $x = -1$, $\frac{dy}{dx} = ?$

$@x=-1: \; -3y - 2 = 1 - 2y$

$-3 = y$

So, pt is $(-1,-3)$

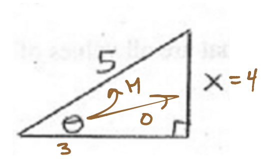
$\frac{d}{dx}: \; 3y + 3x\frac{dy}{dx} + 2 = 0 - 2\frac{dy}{dx}$

$3x\frac{dy}{dx} + 2\frac{dy}{dx} = -2 - 3y$

$\frac{dy}{dx}(3x+2) = -2 - 3y$

$\frac{dy}{dx} = \frac{-2-3y}{3x+2}$ or $-\frac{3y+2}{3x+2}$

5. If in the triangle at right, θ decreases by 4 rad/min, at what rate is x changing in units/min when $\boxed{x = 4}$?



$\frac{d\theta}{dt} = -4$ $\qquad$ $\frac{dx}{dt} = ?$

$\sin\theta = \frac{x}{5}$

$\frac{d}{dt}: \cos\theta \cdot \frac{d\theta}{dt} = \frac{1}{5} \cdot \frac{dx}{dt}$

when $x=4$: $\left(\frac{3}{5}\right)(-4) = \frac{1}{5}\frac{dx}{dt}$

$\frac{dx}{dt} = -12$ rad/min*

* Thanks, Col. Bird, 5/10/19

6. Write an integral equation which gives the length of the path described by the parametric equations $x = \cos^3 t$ and $y = \sin^3 t$ for $0 \le t \le \frac{\pi}{2}$.

$\mathcal{L}$ = Distance Travelled $= \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$

$x' = 3\cos^2 t \cdot (-\sin t)$ $\quad$ $y' = 3\sin^2 t \cdot (\cos t)$

$$\mathcal{L} = \int_0^{\pi/2} \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cdot \cos t)^2}\, dt$$

7. If f is a function such that $f'(x)=\sin\left(x^2\right)$, then the coefficient of x^7 in the Taylor series for $f(x)$ about $x=0$ is?

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$f'(x) = \sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$$

* Thanks, Joy B!

$$f(x) = C + \frac{x^3}{3} - \underbrace{\frac{x^7}{7\cdot 3!}} + \frac{x^{11}}{11\cdot 5!} - \cdots$$

coefficient of x^7 is $-\frac{1}{7\cdot 3!} = \frac{-1}{42}$

8. If $\dfrac{dy}{dx} = y\sec^2 x$ and $y=5$ when $x=0$, then $y=?$

$$\frac{1}{y}\,dy = \sec^2 x\,dx$$

$$\ln|y| = \tan x + C$$

$$e^{\ln|y|} = e^{(\tan x + C)}$$

$$|y| = e^{\tan x}\cdot e^{C}, \quad \text{Let } C = e^{C}$$

$$y = Ce^{\tan x} \quad \text{(gen solution)}$$

$@(0,5)$: $5 = Ce^{\tan 0}$

$5 = Ce^{0}$

$C = 5$

So, $y = 5e^{\tan x}$

9. (a) $\int \tan^2 x dx =$

$\int (\sec^2 x - 1) dx$ Baby PID

$\tan x - x + C$

*thanks, Alain D!

(b) $\int \sin^2 x dx =$

$\int \frac{1}{2}(1 - \cos 2x)\, dx$

$\frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C$

$\frac{1}{4}\left[2x - \sin 2x\right] + C$

or

$\frac{1}{2}x - \frac{1}{4}\sin 2x + C$

10. $\lim_{h\to 0} \frac{\cos\left(\frac{\pi}{2}+h\right) - \cos\left(\frac{\pi}{2}\right)}{h} =$ $f'\left(\frac{\pi}{2}\right)$, for $f(x) = \cos x$

So, $f'(x) = -\sin x$

$f'\left(\frac{\pi}{2}\right) = -\sin\frac{\pi}{2}$

$= -1$

or $\frac{0}{0}$, by L'Hôpital's Rule

$\lim_{h\to 0} \frac{-\sin\left(\frac{\pi}{2}+h\right) - 0}{1}$

$-\sin\left(\frac{\pi}{2}\right)$

-1

11. The coefficient of x^3 in the Taylor series for e^{2x} about $x = 0$ is?

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{2x} = 1 + (2x) + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots$$

$$= 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \cdots$$

So, Coefficient of x^3 is $\frac{8}{3!} = \frac{8}{3\cdot 2\cdot 1} = \frac{4}{3}$

12. What are all values of x for which the series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n3^n}$ converge?

Ratio Test

$$\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)\,3^{n+1}} \cdot \frac{n\cdot 3^n}{(x-2)^n} \right|$$

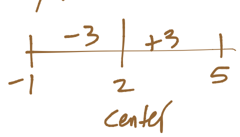
$$\lim_{n\to\infty} \left| \frac{\cancel{(x-2)^n}\cdot(x-2)^1\cdot n\cdot\cancel{3^n}}{(n+1)\cancel{3^n}\cdot 3\,\cancel{(x-2)^n}} \right|$$

$$\lim_{n\to\infty} \left| \frac{(x-2)n^1}{3(n^1+1)} \right|$$

$$\left| \frac{x-2}{3} \right| < 1$$

$|x-c|<R$ $\quad |x-2| < 3$

So, Radius $R = 3$



Test endpoints

$x = -1$: $\sum \frac{(-3)^n}{n\cdot 3^n}$ $\quad (-1)^n\cdot 3^n$

$$\sum \frac{(-1)^n 3^n}{n\,3^n}$$

$$\sum \frac{(-1)^n}{n}$$

Convergent (Conditionally) Harmonic p-series

$x = 5$: $\sum \frac{3^n}{n\cdot 3^n}$

$$\sum \frac{1}{n}$$

Divergent harmonic p-series

So, Interval of Convergence is $[-1, 5)$

B 13. Which of the following converge?

$$\cancel{\text{I.}}\ \sum_{n=1}^{\infty}\frac{n}{n+2} \qquad \text{II. } \sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n} \qquad \cancel{\text{III.}}\ \sum_{n=1}^{\infty}\frac{1}{n}$$

(A) None (B) II only (C) III only (D) I and II only (E) I and III only

I. $\lim_{n\to\infty}\frac{n}{n+2}=1\neq 0$
Diverges by n^{th} term test

II. $\cos(n\pi)=(-1)^n$
Cond. Convergent harmonic series

III. Divergent harmonic Series

14. The Taylor series about $x=5$ for a certain function f converges to $f(x)$ for all x in its interval of convergence. The nth derivative of f at $x=5$ is given by

$$f^{(n)}(5)=\frac{(-1)^n n!}{2^n(n+2)} \text{ and } f(5)=\frac{1}{2}$$

Show that the sixth-degree Taylor polynomial for f about $x=5$ approximates $f(6)$ with an error less than $\frac{1}{1000}$.

This is an alternating series

So, $$|T_6(6)-f(6)| \leq \left|\underbrace{\frac{f^{(7)}(6)}{7!}(6-5)^7}_{\text{1st unused term @ } x=6}\right|$$

$$=\left|\frac{7!}{2^7(7+2)7!}\cdot 1^7\right|$$

$$=\frac{1}{9\cdot 2^7}$$

$$=\frac{1}{1152}<\frac{1}{1000}$$

15. The population $P(t)$ of unicorns in a forest satisfies the logistic differential equation $\frac{dP}{dt} = 3P - \frac{P^2}{6000}$.

(a) If $P(0) = 4000$, what is $\lim_{t\to\infty} P(t)$? Is the solution curve increasing or decreasing? Justify your answer.

(b) If $P(0) = 10{,}000$, what is $\lim_{t\to\infty} P(t)$? Is the solution curve increasing or decreasing? Justify your answer.

(c) If $P(0) = 20{,}000$, what is $\lim_{t\to\infty} P(t)$? Is the solution curve increasing or decreasing? Justify your answer.

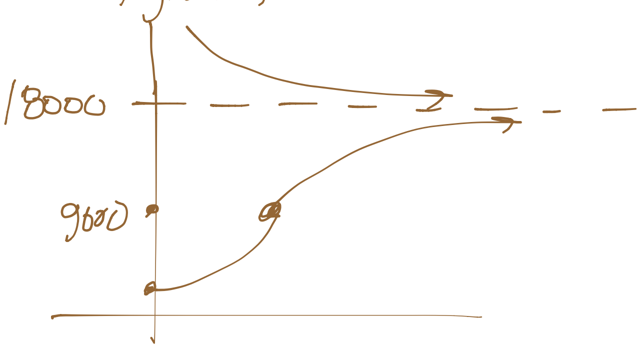
(d) If $P(0) = 4000$, what is the population when it is growing the fastest? Where is the solution curve concave up? Concave down? Justify your answer.

$$\frac{dP}{dt} = 3P - \frac{P^2}{6000}$$

$$\frac{dP}{dt} = \frac{1}{6000} P\,(18{,}000 - P)$$

So, $k = \frac{1}{6000}$, $L = 18{,}000$

P grows fastest when $L = \frac{18000}{2} = 9000$



Solution curves are concave up for $0 < P(t) < 9000$, & $18000 < P(t) < 27000$

So, for $P(0) = 4000$, $P(t)$ is concave down for $9000 < P(t) < 18000$

(a) $\lim_{t\to\infty} P(t) = 18000$, Increasing since $P(0) = 4000 < 18000$

(b) $\lim_{t\to\infty} P(t) = 18000$, Increasing since $P(0) = 10000 < 18000$

(c) $\lim_{t\to\infty} P(t) = 18000$, Decreasing since $P(0) = 20000 > 18000$

(d) $P(t) = 9000$ when population grows fastest. At this time,

$$\frac{dP}{dt} = \frac{9000}{6000}(18000 - 9000) = \frac{3}{2}(9000) = 13500 \text{ unicorns/time}$$

(Xtra info !!)

16. (Calculator Permitted) Given $\overset{y1=}{\frac{dx}{dt}} = \cos\left(t^3\right)$ and $\overset{y2=}{\frac{dy}{dt}} = 3\sin\left(t^2\right)$ for $0 \le t \le 3$. At time $t = 2$, the object at position $(\overset{x}{4},\overset{y}{5})$.

$x'(t) =$ $\quad y'(t) =$

(a) Find the speed of the object at time $t = 2$.

(b) Find the total distance traveled by the object over the time interval $0 \le t \le 1$.

(c) Find the position of the object at time $t = 3$.

(a) Speed $= \sqrt{[x'(2)]^2 + [y'(2)]^2}$

$= 2.275$

(b) Distance $= \mathcal{L} = \int_0^1 \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$

$= 1.458$

(c) $\vec{s}(3) = \langle x(3), y(3) \rangle$

$= \left\langle 4 + \int_2^3 x'(t)\,dt,\ 5 + \int_2^3 y'(t)\,dt \right\rangle$

$= \langle 3.953, 4.906 \rangle$

17. (Calculator Permitted) Given $f(x) = \frac{1}{3} + \frac{2x}{9} + \frac{3x^2}{27} + \cdots + \frac{(n+1)}{3^{n+1}} x^n + \cdots$

(a) $\lim_{x \to 0} \frac{f(x) - \frac{1}{3}}{x}$.

$f'(x) = \frac{2}{9} + \frac{6}{27}x + \cdots + \frac{n(n+1)}{3^{n+1}} x^{n-1} + \cdots$

(b) Write the first three nonzero terms and the general term for the infinite series that represents $\int_0^1 f(x)dx$.

(c) Find the sum of the series found in part (b).

(a) $\lim_{x \to 0} \frac{f(x) - \frac{1}{3}}{x}$ $\quad \frac{0}{0}$

L'Hop: $\lim_{x \to 0} \frac{f'(x)}{1} = \frac{2}{9}$

(b) $\int_0^1 f(x)dx = \left. \frac{1}{3}x + \frac{1}{9}x^2 + \frac{1}{27}x^3 + \cdots + \frac{1}{3^{n+1}} x^{n+1} + \cdots \right|_0^1$

(c)

$$= \left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots + \frac{1}{3^{n+1}} + \cdots\right) - (0)$$

$$= \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots \frac{1}{3^{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{3^{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{1}{3} \cdot \left(\frac{1}{3}\right)^n$$ (Geometric Series) $|r| = \left|\frac{1}{3}\right| = \frac{1}{3} < 1$

$$= \frac{1/3}{1 - \frac{1}{3}}$$

1st term; r

$$= \frac{\frac{1}{3}}{\frac{2}{3}}$$

$$= \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$$

18. (Calculator Permitted) The function f has derivatives of all orders for all real numbers x. Assume that $f(2)=5$, $f'(2)=-3$, $f''(2)=4$, $f'''(2)=-1$, and $\left|f^{(4)}(x)\right| \le 3$ for all x in $[2, 2.2]$.

(a) Write the third-degree Taylor polynomial for f about $x=2$.

(b) Use your answer to (a) to approximate $f(2.15)$. Give your answer correct to five decimal places.

(c) Use the Lagrange error bound on the approximation of $f(2.15)$ to explain why $f(2.15) \neq 4.7$.

(a) $f(x) \approx T_3(x) = 5 - 3(x-2) + \frac{4}{2!}(x-2)^2 - \frac{1}{3!}(x-2)^3$

(b) $f(2.15) \approx T_3(2.15) = 4.59443 = A$ (store)

(c) $\left| f(2.15) - T_3(2.15) \right| = \left| \frac{f^{(4)}(z)}{4!} (2.15-2)^4 \right|$

$\le \left| \frac{3}{24} (.15)^4 \right| = 0.0000632812 5 = B$ (store)

$\frac{15}{100} = \frac{3}{20}$

So $f(2.15) \in [T_3(2.15) - B, T_3(2.15) + B]$

$f(2.15) \in [A - B, A + B] = [4.594374219, 4.594500781]$

So, $f(2.15) \neq 4.7$ since 4.7 is outside the above interval.