1. D
2. A
3. B
4. B
5. A
6. C
7. D
8. B
9. E
10. C

4

BC Review 04, No calculator (unless specified otherwise)

1. Let f be a twice differentiable function such that $f(1)=2$ and $f(3)=7$. Which of the following must be true for the function f on the interval $1 \le x \le 3$?

   I. The average rate of change of f is $\frac{5}{2}$.

   $\frac{7-2}{3-1} = \frac{5}{2}$

   II. The average value of f is $\frac{9}{2}$.

   $\frac{\int_1^3 f(x)\,dx}{3-1}$ $\quad \frac{F(x)\big|_1^3}{2}$ $\quad \frac{1}{2}[F(3)-F(1)]$

   III. The average value of f' is $\frac{5}{2}$.

   (A) None (B) I only (C) III only (D) I and III only (E) II and III only

   $\frac{\int_1^3 f'(x)\,dx}{3-1} = \frac{f(x)\big|_1^3}{3-1} = \frac{f(3)-f(1)}{3-1}$

2. $\displaystyle\int \frac{dx}{(x-1)(x+3)} =$

   (A) $\frac{1}{4}\ln\left|\frac{x-1}{x+3}\right|+C$ (B) $\frac{1}{4}\ln\left|\frac{x+3}{x-1}\right|+C$ (C) $\frac{1}{2}\ln\left|(x-1)(x+3)\right|+C$

   (D) $\frac{1}{2}\ln\left|\frac{2x+2}{(x-1)(x+3)}\right|+C$ (E) $\ln\left|(x-1)(x+3)\right|+C$

3. The base of a solid is the region in the first quadrant enclosed by the graphs of $y=2-x$ and the coordinate axes. If every cross section of the solid perpendicular to the y-axis is a square, the volume of the solid is given by

(A) $\pi\int_0^2 (2-y)^2\,dy$ (B) $\int_0^2 (2-y)^2\,dy$ (C) $\pi\int_0^{\sqrt{2}} \left(2-x^2\right)^2 dx$ (D) $\int_0^{\sqrt{2}} \left(2-x^2\right)^2 dx$ (E) $\int_0^{\sqrt{2}} \left(2-x^2\right) dx$

dy

4. $\lim_{h \to 0} \dfrac{\tan\left(3(x+h)\right) - \tan(3x)}{h} =$

(A) 0 (B) $3\sec^2(3x)$ (C) $\sec^2(3x)$ (D) $3\cot(3x)$ (E) nonexistent

$y = \tan 3x$

$y' = 3\sec^2 3x$

5. (Calculator Permitted) Let $F(x) = \boxed{\cos(2x) + e^{-x}}$. For what value of x on the interval $[0,3]$ will F have the same instantaneous rate of change as the average rate of change of F over the interval?

(A) 1.542 (B) 1.610 (C) 1.678 (D) 1.746 (E) 1.814

MVT $F'(x) = \boxed{\dfrac{F(3) - F(0)}{3-0}}$

$-2\sin(2x) - e^{-x} - \dfrac{\cos 6 + e^{-3} - (1+1)}{3} = 0$

6. Let f be a differentiable function such that $f(3) = 2$ and $f'(3) = 5$. If the tangent line to the graph of f at $x = 3$ is used to find an approximation to a zero of f, that approximation is

(A) 0.4 (B) 0.5 (C) 2.6 (D) 3.4 (E) 5.5

zero = x-int

$y = 2 + 5(x-3) = T_1(x)$

$0 = 2 + 5x - 15$

$13 = 5x$

$x = \frac{13}{5} \cdot \frac{2}{2} = \frac{26}{10}$

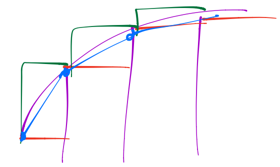
$13.0 \div 5 = 2.6$

7. (Calculator Permitted) If $f'(x) = \dfrac{x^2}{1+x^5}$ and $f(1) = 3$, then $f(4) =$

(A) 2.988 (B) 3 (C) 3.016 (D) 3.376 (E) 3.629

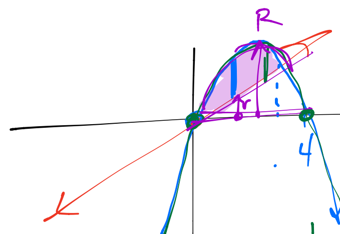
$\frac{1}{2}\left[\frac{1}{2}(3+3) + \frac{1}{2}(3+5)\right.$

x	0	0.5	1.0	1.5	2.0
$f(x)$	3	3	5	8	13



8. A table of values for a continuous function f is shown above. If four equal subintervals of $[0,2]$ are used, which of the following is the trapezoidal approximation of $\int_0^2 f(x)\,dx$? $\approx \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left[3+2(3)+2(5)+2(8)+13\right]$

(A) 8 (B) 12 (C) 16 (D) 24 (E) 32



$V=\pi\int_0^3\left[(4x-x^2-0)^2-(x-0)^2\right]dx$

$4x-x^2-x=0$

$V=2\pi\int_0^3 (x)(4x-x^2-x)\,dx$

9. When the region enclosed by graphs of $y=x$ and $y=4x-x^2$ is revolved about the y-axis, the volume of the solid generated is given by

$4x-x^2=0$

$x(4-x)=0$

$x=0 \quad x=4$

(A) $\pi\int_0^3 \left(x^3-3x^2\right)dx$ (B) $\pi\int_0^3 \left(x^2-\left(4x-x^2\right)^2\right)dx$ (C) $\pi\int_0^3 \left(3x-x^2\right)^2 dx$

(D) $2\pi\int_0^3 \left(x^3-3x^2\right)dx$ (E) $2\pi\int_0^3 \left(3x^2-x^3\right)dx$

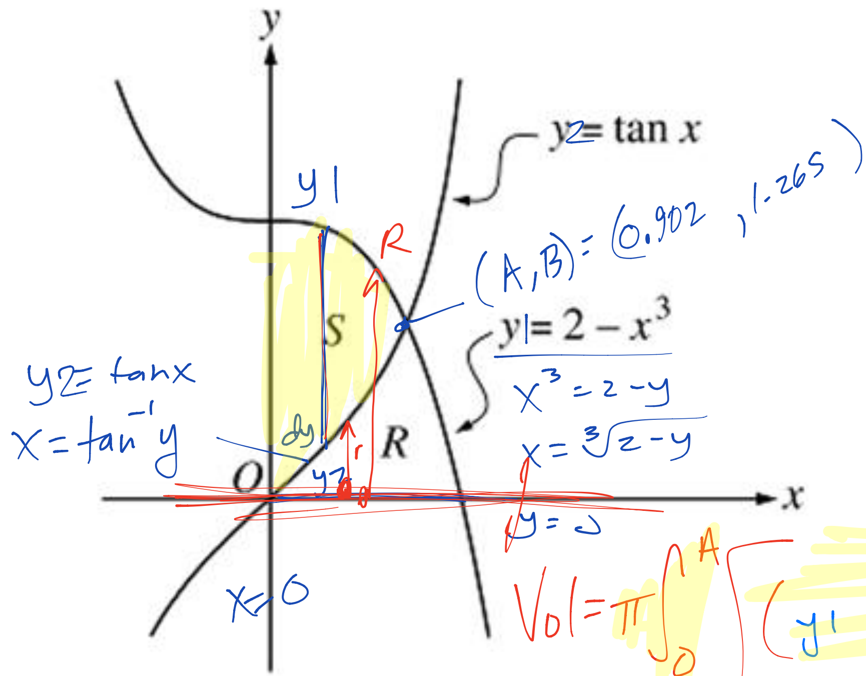
$x(3x-x^2)$

$3x^2-x^3$

Time (sec)	0	10	25	37	46	60
Rate (gal/sec)	500	400	350	280	200	180

10. (Calculator Permitted) The table above gives the values for the rate (in gal/sec) at which water flowed into a lake, with readings taken at specific times. A right Riemann sum, with five subintervals indicated by the date in the table, is used to estimate the total amount of water that flowed into the lake during the time period $0 \le t \le 60$. What is this estimate?

(A) 1,910 gal (B) 14,100 gal (C) 16,930 gal (D) 18,725 gal (E) 20,520 gal



$$Vol = \pi \int_0^A \left[(y1 - 0)^2 - (y2 - 0)^2 \right] dx$$

$= 8.331$

11. (Calculator Permitted), (2001, AB-1) Let R and S be the regions in the first quadrant shown in the figure above. The region R is bounded by the x-axis and the graphs of $y1 = 2 - x^3$ and $y2 = \tan x$. The region S is bounded by the y-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$.

(a) Find the area of R.

$$Area = \int_0^B \left(\underbrace{\sqrt[3]{2-y}}_{R} - \underbrace{\tan^{-1} y}_{L} \right) dy$$

$= 0.729$

(b) Find the area of S.

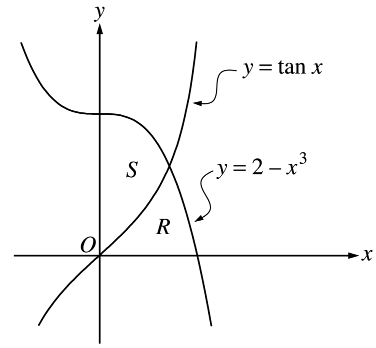
$$Area = \int_0^A (y1 - y2)\, dx$$

$= 1.160$

(c) Find the volume of the solid generated when S is revolved about the x-axis.

Question 1

Let R and S be the regions in the first quadrant shown in the figure above. The region R is bounded by the x-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$. The region S is bounded by the y-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$.



(a) Find the area of R.

(b) Find the area of S.

(c) Find the volume of the solid generated when S is revolved about the x-axis.

Point of intersection

$2 - x^3 = \tan x$ at $(A, B) = (0.902155, 1.265751)$

(a) Area $R = \int_0^A \tan x\,dx + \int_A^{\sqrt[3]{2}} (2 - x^3)\,dx = 0.729$

or

Area $R = \int_0^B \left((2-y)^{1/3} - \tan^{-1} y\right) dy = 0.729$

or

Area $R = \int_0^{\sqrt[3]{2}} (2 - x^3)\,dx - \int_0^A (2 - x^3 - \tan x)\,dx = 0.729$

3 : { 1 : limits; 1 : integrand; 1 : answer }

(b) Area $S = \int_0^A (2 - x^3 - \tan x)\,dx = 1.160$ or 1.161

or

Area $S = \int_0^B \tan^{-1} y\,dy + \int_B^2 (2-y)^{1/3}\,dy = 1.160$ or 1.161

or

Area S

$= \int_0^2 (2-y)^{1/3}\,dy - \int_0^B \left((2-y)^{1/3} - \tan^{-1} y\right) dy$

$= 1.160$ or 1.161

3 : { 1 : limits; 1 : integrand; 1 : answer }

(c) Volume $= \pi \int_0^A \left((2 - x^3)^2 - \tan^2 x\right) dx$

$= 2.652\pi$ or 8.331 or 8.332

3 : { 1 : limits and constant; 1 : integrand; 1 : answer }

12. (2001, AB-6) The function f is differentiable for all real numbers. The point $\left(3,\frac{1}{4}\right)$ is on the graph of $y=f(x)$, and the slope at each point (x,y) on the graph is given by $\frac{dy}{dx}=6y^2-2xy^2$. $=2y^2(3-2x)$

$$\left.\frac{dy}{dx}\right|_{\left(3,\frac{1}{4}\right)} = \frac{6}{16}-\frac{6}{16} = 0$$

(a) Find $\frac{d^2y}{dx^2}$ and evaluate it at the point $\left(3,\frac{1}{4}\right)$.

$$\frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[6y^2-2xy^2\right]$$

$$\frac{d^2y}{dx^2} = 12y\frac{dy}{dx} - 2y^2 - 2x\cdot 2y\frac{dy}{dx}$$

$$\left.\frac{d^2y}{dx^2}\right|_{\left(3,\frac{1}{4}\right)} = (-2)\left(\frac{1}{16}\right) = -\frac{1}{8}$$

(b) Find $y=f(x)$ by solving the differential equation $\frac{dy}{dx}=6y^2-2xy^2$ with the initial condition $f(3)=\frac{1}{4}$.

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{2y^2(3-x)}{y^2}$$

$$\int \frac{1}{y^2}\,dy = \int (6-2x)\,dx \quad ✓$$

$$\int y^{-2}\,dy = \int (6-2x)\,dx$$

$$-y^{-1} = 6x - x^2 + C \quad ✓2 \quad ✓3 \quad ✓4$$

$$y^{-1} = -6x + x^2 + C$$

$$\frac{1}{y} = \frac{x^2-6x+C}{1}$$

gen

$$\boxed{y = \frac{1}{x^2-6x+C}}$$

At $\left(3,\frac{1}{4}\right)$

for $f(3)=\frac{1}{4}$:

$$\frac{1}{4} = \frac{1}{9-18+C} \quad ✓5$$

$$\frac{1}{4} = \frac{1}{C-9}$$

$$C-9=4$$

$$\boxed{C=13}$$

So $y = \frac{1}{x^2-6x+13}$ ✓6

5x − 7?

Question 6

The function f is differentiable for all real numbers. The point $\left(3, \frac{1}{4}\right)$ is on the graph of $y = f(x)$, and the slope at each point (x, y) on the graph is given by $\frac{dy}{dx} = y^2(6 - 2x)$.

(a) Find $\frac{d^2y}{dx^2}$ and evaluate it at the point $\left(3, \frac{1}{4}\right)$.

(b) Find $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = y^2(6 - 2x)$ with the initial condition $f(3) = \frac{1}{4}$.

(a)
$$\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}(6-2x) - 2y^2$$
$$= 2y^3(6-2x)^2 - 2y^2$$

$$\left.\frac{d^2y}{dx^2}\right|_{\left(3,\frac{1}{4}\right)} = 0 - 2\left(\frac{1}{4}\right)^2 = -\frac{1}{8}$$

$$3: \begin{cases} 2: \dfrac{d^2y}{dx^2} \\ \quad <-2> \text{ product rule or chain rule error} \\ 1: \text{value at } \left(3, \frac{1}{4}\right) \end{cases}$$

(b)
$$\frac{1}{y^2}\,dy = (6-2x)\,dx$$

$$-\frac{1}{y} = 6x - x^2 + C$$

$$-4 = 18 - 9 + C = 9 + C$$

$$C = -13$$

$$y = \frac{1}{x^2 - 6x + 13}$$

$$6: \begin{cases} 1: \text{separates variables} \\ 1: \text{antiderivative of } dy \text{ term} \\ 1: \text{antiderivative of } dx \text{ term} \\ 1: \text{constant of integration} \\ 1: \text{uses initial condition } f(3) = \frac{1}{4} \\ 1: \text{solves for } y \end{cases}$$

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables