1. A
2. D
3. B
4. A
5. A
6. C
7. E
8. E
9. A
10. D

10

BC Review 10   No Calculator Permitted

1.

If $f(x) = (\ln x)^2$, then $f''(\sqrt{e}) =$

(A) $\frac{1}{e}$ (B) $\frac{2}{e}$ (C) $\frac{1}{2\sqrt{e}}$ (D) $\frac{1}{\sqrt{e}}$ (E) $\frac{2}{\sqrt{e}}$

2.

What are all values of x for which the series $\sum_{n=1}^{\infty} \left(\frac{2}{x^2 + 1} \right)^n$ converges?

(A) $\cancel{-1 < x < 1}$

(B) $x > 1$ only

(C) $x \geq 1$ only

(D) $x < -1$ and $x > 1$ only

(E) $x \leq -1$ and $x \geq 1$

$\Sigma \left(\frac{2}{2} \right)^n$

1^n

$\Sigma 1$

3.

Let h be a differentiable function, and let f be the function defined by $f(x) = h(x^2 - 3)$. Which of the following is equal to $f'(2)$?

(A) $h'(1)$ (B) $4h'(1)$ (C) $4h'(2)$ (D) $h'(4)$ (E) $4h'(4)$

$f'(x) = h'(x^2-3)(2x)$

$f'(2) = h'(1)(4)$

4.

In the xy-plane, the line $x + y = k$, where k is a constant, is tangent to the graph of $y = x^2 + 3x + 1$. What is the value of k ?

$y = -x + k$

(A) -3 (B) -2 (C) -1 (D) 0 (E) 1

$\frac{3/2}{7/2} = \frac{3}{7}$ $\frac{-14}{-7} = 2$

5.

$$\int \frac{7x}{(2x-3)(x+2)}\,dx =$$

$\int \left(\frac{3}{(2x-3)} + \frac{2}{(x+2)}\right)$

(A) $\frac{3}{2}\ln|2x-3| + 2\ln|x+2| + C$ $3\left(\frac{1}{2}\right)\ln|2x-3| + 2\ln|x+2|$

(B) $3\ln|2x-3| + 2\ln|x+2| + C$

(C) $3\ln|2x-3| - 2\ln|x+2| + C$

(D) $-\frac{6}{(2x-3)^2} - \frac{2}{(x+2)^2} + C$

(E) $-\frac{3}{(2x-3)^2} - \frac{2}{(x+2)^2} + C$

6.

What is the sum of the series $1 + \ln 2 + \frac{(\ln 2)^2}{2!} + \cdots + \frac{(\ln 2)^n}{n!} + \cdots$?

(A) $\ln 2$

(B) $\ln(1 + \ln 2)$

(C) 2

(D) e^2

(E) The series diverges.

$e^{\ln 2} = 2$

7.

x	0	1
$f(x)$	2	4
$f'(x)$	6	–3
$g(x)$	–4	3
$g'(x)$	2	–1

$\cancel{F \quad g''}$

The table above gives values of f, f', g, and g' for selected values of x. If $\int_0^1 f'(x)g(x)\,dx = 5$, then $\int_0^1 f(x)g'(x)\,dx =$

(A) –14 (B) –13 (C) –2 (D) 7 (E) 15

$f' \quad g$

u	v	+/–
f	g'	+
f'	g	–

$fg - \int f'g\,dx$

$fg\Big|_0^1 \quad -5$

$\boxed{f(1)g(1) - f(0)g(0)}$ -5

8.

If $f(x) = x\sin(2x)$, which of the following is the Taylor series for f about $x = 0$?

(A) $x - \dfrac{x^3}{2!} + \dfrac{x^5}{4!} - \dfrac{x^7}{6!} + \cdots$

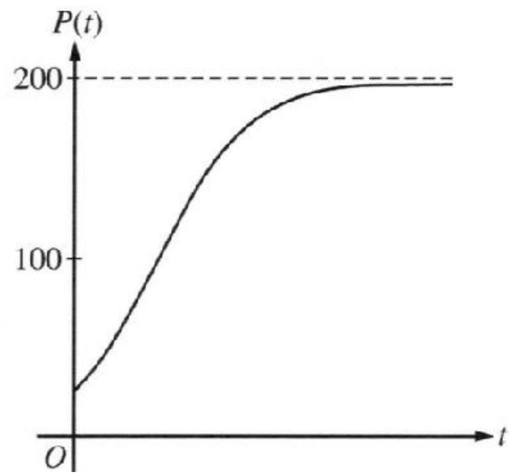
(B) $x - \dfrac{4x^3}{2!} + \dfrac{16x^5}{4!} - \dfrac{64x^7}{6!} + \cdots$

(C) $2x - \dfrac{8x^3}{3!} + \dfrac{32x^5}{5!} - \dfrac{128x^7}{7!} + \cdots$

(D) $2x^2 - \dfrac{2x^4}{3!} + \dfrac{2x^6}{5!} - \dfrac{2x^8}{7!} + \cdots$

(E) $2x^2 - \dfrac{8x^4}{3!} + \dfrac{32x^6}{5!} - \dfrac{128x^8}{7!} + \cdots$

9.



Which of the following differential equations for a population P could model the logistic growth shown in the figure above?

(A) $\dfrac{dP}{dt} = 0.2P - 0.001P^2$

(B) $\dfrac{dP}{dt} = 0.1P - 0.001P^2$

(C) $\dfrac{dP}{dt} = 0.2P^2 - 0.001P$

(D) $\dfrac{dP}{dt} = 0.1P^2 - 0.001P$

(E) $\dfrac{dP}{dt} = 0.1P^2 + 0.001P$

10.

In the xy-plane, a particle moves along the parabola $y = x^2 - x$ with a constant speed of $2\sqrt{10}$ units per second. If $\frac{dx}{dt} > 0$, what is the value of $\frac{dy}{dt}$ when the particle is at the point $(2, 2)$?

(A) $\frac{2}{3}$ (B) $\frac{2\sqrt{10}}{3}$ (C) 3 (D) 6 (E) $6\sqrt{10}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$

$$2x-1 = \frac{y'(t)}{x'(t)}$$

at $x=2$: $3x'(t) = y'(t)$

$$\boxed{x'(t) = \tfrac{1}{3}y'(t)}$$

$$2\sqrt{10} = \sqrt{(y'(t))^2 + (x'(t))^2}$$

$$40 = (y'(t))^2 + (x'(t))^2$$

$$40 = (y'(t))^2 + \tfrac{1}{9}(y'(t))^2$$

$$40 = \tfrac{10}{9}(y'(t))^2$$

$$36 = (y'(t))^2$$

$$\boxed{y'(t) = 6}$$

2 1

$g'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 + \cdots + \frac{(-1)^n(2n+1)x^{2n}}{2n+3} + \cdots$

11. (2012, BC-6)

The function g has derivatives of all orders, and the Maclaurin series for g is

$$g(x) = \sum_{n=0}^{\infty} \boxed{(-1)^n \frac{x^{2n+1}}{2n+3}} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots \quad \text{Error} \leq \left| \frac{\left(\frac{1}{2}\right)^5}{7} \right| = \frac{1}{32 \cdot 7} = \frac{1}{224} < \frac{1}{200}$$

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\boxed{\frac{17}{120}}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\boxed{\frac{1}{200}}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

(a) $\lim_{n\to\infty} \left| \frac{x^{2n+3}}{2n+5} \cdot \frac{2n+3}{x^{2n+1}} \right|$

$\lim_{n\to\infty} \left| \frac{x^2(2n+3)}{(2n+5)} \right|$

$|x-c| < 1 \qquad |x^2| < 1$

Interval $\rightarrow$ $\boxed{-1 \leq x \leq 1}$

$\boxed{[-1, 1]}$

Question 6

The function g has derivatives of all orders, and the Maclaurin series for g is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots.$

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

(a) $\left|\frac{x^{2n+3}}{2n+5} \cdot \frac{2n+3}{x^{2n+1}}\right| = \left(\frac{2n+3}{2n+5}\right) \cdot x^2$

$\lim_{n\to\infty}\left(\frac{2n+3}{2n+5}\right) \cdot x^2 = x^2$

$x^2 < 1 \Rightarrow -1 < x < 1$

The series converges when $-1 < x < 1$.

When $x = -1$, the series is $-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

This series converges by the Alternating Series Test.

When $x = 1$, the series is $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$

This series converges by the Alternating Series Test.

Therefore, the interval of convergence is $-1 \le x \le 1$.

5 : { 1 : sets up ratio
1 : computes limit of ratio
1 : identifies interior of interval of convergence
1 : considers both endpoints
1 : analysis and interval of convergence

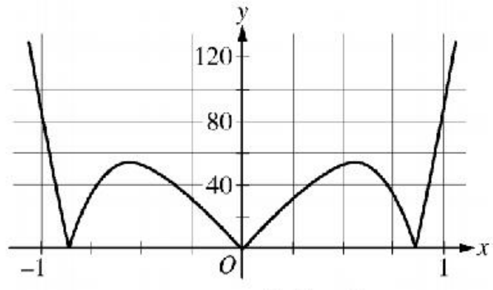
(b) $\left|g\left(\frac{1}{2}\right) - \frac{17}{120}\right| < \frac{\left(\frac{1}{2}\right)^5}{7} = \frac{1}{224} < \frac{1}{200}$

2 : { 1 : uses the third term as an error bound
1 : error bound

(c) $g'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 + \cdots + (-1)^n\left(\frac{2n+1}{2n+3}\right)x^{2n} + \cdots$

2 : { 1 : first three terms
1 : general term

12. (2011, BC-6)



Graph of $y = \left|f^{(5)}(x)\right|$

Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

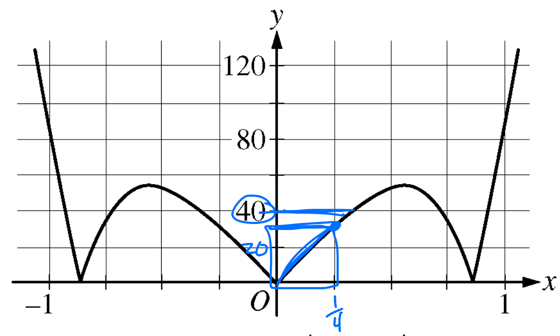
(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

Question 6

Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $\boxed{f^{(6)}(0)}$. $\frac{f^{(n)}(0)}{n!}x^n$

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

$256 \cdot 10 = 2560$ $\quad 256 \cdot 2$

$$\text{Error} = R_4 = \left|\frac{f^{(5)}(z)}{5!}(x-0)^5\right| \le \left|\frac{40}{5!}\left(\frac{1}{4}\right)^5\right| = \frac{\cancel{40}\,10}{\cancel{5}\cdot 4\cdot 3\cdot 2\cdot 1}\left(\frac{1}{4\cdot 4\cdot 4\cdot 4\cdot \cancel{4}}\right) = \frac{1}{16^2\cdot 12}$$

(a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\sin\left(x^2\right) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$

3 : { 1 : series for $\sin x$; 2 : series for $\sin\left(x^2\right)$ }

$$f(x) = 1 + \left(x^2 - \frac{x^2}{2!}\right) + \left(\frac{x^4}{4!}\right) - \left(\frac{x^6}{6!} + \frac{x^6}{3!}\right) + \cdots$$

(b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121x^6}{6!} + \cdots$

3 : { 1 : series for $\cos x$; 2 : series for $f(x)$ }

$$\left(-\frac{1}{6!} - \frac{1}{3!}\cdot\frac{4\cdot 5\cdot 6}{4\cdot 5\cdot 6}\right)x^6$$

$$\frac{-1-4\cdot 5\cdot 6}{6!}x^6$$

so $f^{(6)}(0) = \boxed{-1-4\cdot 5\cdot 6} = \boxed{-121}$

(c) $\frac{f^{(6)}(0)}{6!}$ is the coefficient of x^6 in the Taylor series for f about $x = 0$. Therefore $f^{(6)}(0) = -121$.

1 : answer

(d) The graph of $y = \left|f^{(5)}(x)\right|$ indicates that $\max\limits_{0 \le x \le \frac{1}{4}}\left|f^{(5)}(x)\right| < 40$.

Therefore

$$\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| \le \frac{\max\limits_{0\le x\le\frac{1}{4}}\left|f^{(5)}(x)\right|}{5!}\cdot\left(\frac{1}{4}\right)^5 < \frac{40}{120\cdot 4^5} = \frac{1}{3072} < \frac{1}{3000}.$$

2 : { 1 : form of the error bound; 1 : analysis }