Practice Test 2 — BC

Section I: Multiple-Choice Questions

Section IA

Time: 55 Minutes

28 Questions

Directions: The 28 questions that follow in Section IA of the exam should be solved using the space available for scratchwork. Select the best of the given choices and fill in the corresponding oval on the answer sheet. Material written in the test booklet will not be graded or awarded credit. Fifty-five minutes are allowed for Section IA. *No calculator of any type may be used in this section of the test.*

Notes: *(1) For this test, ln* x *denotes the natural logarithm of* x *(that is, logarithm of the base* e*). (2) The domain of all functions is assumed to be the set of real numbers* x *for which* f(x) *is a real number, unless a different domain is specified.*

1. If $g(t)=\frac{e^{\ln t}}{t}$, then $g(e)=$

A. $\frac{e}{2}$

B. e

C. 1

D. $\frac{1}{e}$

E. 0

2. A particle moves along the curve $xy = 12$. If $y = 3$ and $\frac{dx}{dt} = 2$, what is the value of $\frac{dy}{dt}$?

A. $\frac{-3}{2}$

B. $\frac{-2}{3}$

C. $\frac{2}{3}$

D. $\frac{3}{2}$

E. 6

3. What is $\lim_{\theta \to 0}\left(\frac{3\theta}{\tan\theta}\right)$?

A. $-\infty$

B. 0

C. 1

D. 3

E. $+\infty$

4. Evaluate $\lim_{x \to \infty} e^{-x} \cdot \ln x$.

A. –1

B. $\frac{1}{e}$

C. 1

D. 0

E. does not exist

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5. Given that $f(x)=\begin{cases}3-4x & \text{for } x\le 0\\ x^2 & \text{for } 0<x<2\\ 3x-4 & \text{for } x\ge 2\end{cases}$

Find $\lim_{x\to 2^-} f(x)$.

A. –5

B. 2

C. 4

D. 6

E. The limit does not exist.

6. If $x = t^2 - 1$ and $y = t^2$, then $\frac{d^2y}{dx^2} =$

A. 1

B. $\frac{3}{4t}$

C. $\frac{-2}{9t^4}$

D. $\frac{9t^4}{2}$

E. $\frac{t^4}{\left(t^{3-1}\right)^2}$

7. $\frac{d}{dx}\left[\frac{\cos x}{\sin x+1}\right]=$

A. $\sin x + 1$

B. $-\csc^2 x - \sin x$

C. $-\csc^2 x + 1$

D. $\frac{-\sin x}{\sin x+1}$

E. $\frac{-1}{1+\sin x}$

8. If $y=\sqrt{\ln 3x}$, then $y'=$

A. $\frac{1}{3x\sqrt{\ln 3x}}$

B. $\frac{1}{2x\sqrt{\ln 3x}}$

C. $\frac{1}{6x\sqrt{\ln 3x}}$

D. $\frac{1}{2\sqrt{\ln 3x}}$

E. $\frac{3}{2x\sqrt{\ln 3x}}$

9. $\int x \sin x \, dx =$

A. $-x^2 \cos x + c$

B. $-x\cos x + \sin x + c$

C. $\sin x + x\cos x + c$

D. $x \sin x - \cos x + c$

E. $\cos x + c$

10. If $y = \arctan(3x)$, then $\left.\frac{dy}{dx}\right|_{x=1} =$

A. $\frac{1}{10}$

B. $\frac{1}{4}$

C. $\frac{3}{10}$

D. $\frac{1}{3}$

E. $\frac{3}{4}$

11. $\frac{d}{dy}\left[\log_3 4y\right]=$

A. $\frac{1}{4y\ln 3}$

B. $\frac{\ln 3}{y}$

C. $\frac{4}{y\ln 3}$

D. $\frac{1}{y}$

E. $\frac{1}{y\ln 3}$

12. The graph of $y = e^{-x^2}$ has a point of inflection at

A. $x = 0$

B. $x = \pm 2$

C. $x = \pm\sqrt{2}$

D. $x = \dfrac{\pm\sqrt{2}}{2}$

E. The graph has no points of inflection.

13. The radius of a circle is increasing at the rate of 3 meters per second. Find the rate, in square meters per second, at which the area of the circle is changing when the area is 16π m^2.

A. 8π m^2/s

B. 12π m^2/s

C. 24π m^2/s

D. 96π m^2/s

E. $96\pi^2$ m^2/s

14. A particle moves along a horizontal path such that its position at any time $t\,(t \geq 0)$ is given $s(t) = t^3 - 4t^2 + 4t + 5$. The particle is moving right for

A. $t > 2$ only

B. $0 < t < \frac{2}{3}$ only

C. $\frac{2}{3} < t < 2$

D. $0 < t < \frac{2}{3}$ or $t > 2$

E. $t > \frac{2}{3}$

15. Find the equation of the line that is tangent to the curve $xy - x + y = 2$ at the point where $x = 0$.

A. $y = -x$

B. $y = \frac{1}{2}x + 2$

C. $y = x + 2$

D. $y = 2$

E. $y = -x + 2$

16. $\int 3xe^{2x}\,dx =$

A. $3xe^{2x}\,e - e^{2x} + C$

B. $6xe^{2x} - 4e^{2x} + \text{C}$

C. $xe^{2x} - 3e^{2x} + C$

D. $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$

E. $\frac{3}{2}xe^{2x} - \frac{3}{4}e^{2x} + C$

17. $\int \cot 3\phi\, d\phi =$

A. $\frac{-1}{3}\ln\left|\cos 3\phi\right| + C$

B. $\frac{-1}{3}\ln\left|\csc 3\phi + \cot 3\phi\right| + C$

C. $-3\csc^2 3\phi + C$

D. $\frac{1}{3}\ln\left|\sin 3\phi\right| + C$

E. $3\sec^2 3\phi + C$

18. $\int \dfrac{2x}{\sqrt{x+5}}\,dx =$

A. $x - 10\ln\sqrt{x+5} + C$

B. $\frac{4}{3}\sqrt{x+5}\,(x-10) + C$

C. $\frac{2}{3}\sqrt{x+5}\,(x-10) + C$

D. $\frac{4}{3}\sqrt{x+5}\,(x+20) + C$

E. $2x^2\sqrt{x+5} + C$

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19. The integral $\int_1^2 \frac{dx}{\sqrt[3]{x-1}}$

A. converges to $\frac{3}{2}$

B. converges to 0

C. diverges

D. converges to –1

E. converges to $\frac{-3}{2}$

20. Suppose that $f(x)=\begin{cases} 2x & \text{for } x \le 1 \\ 3x^2-1 & \text{for } x > 1 \end{cases}$

Then $\int_0^2 f(x)\,dx =$

A. 7

B. 6

C. 5

D. 2

E. 1

21. Find the area of the region bounded by the graphs at $y=\frac{x^3+2}{x}$, $y = 0$, $x = 1$, and $x = e$.

A. $\frac{e^2}{2}-\frac{4}{e}-\frac{9}{2}$

B. $\frac{9}{2}-\frac{e^2}{2}-\frac{4}{e}$

C. $2\sqrt{e^9}+\frac{4}{e}-\frac{38}{9}$

D. $\frac{-5-e^3}{3}$

E. $\frac{e^3+5}{3}$

22. The rate of decay of radioactive uranium is proportional to the amount of uranium present at any time t. If there are initially 54 grams of uranium present, and there are 42 grams present after 120 years, this situation can be described by the equation

A. $y=54e^{kt}, k=\frac{1}{120}\ln\frac{7}{9}$

B. $y=54e^{kt}, k=\frac{7}{9}\ln 120$

C. $y=42e^{kt}, k=\frac{1}{120}\ln\frac{7}{9}$

D. $y=42e^{kt}, k=\frac{7}{9}\ln\frac{1}{120}$

E. $y=42e^{kt}, k=120\ln\frac{7}{9}$

23. The average value of tan x on the interval from $x = 0$ to $x = \pi/3$ is

A. $\ln\frac{1}{2}$

B. $\frac{3}{\pi}\ln 2$

C. ln 2

D. $\frac{\sqrt{3}}{2}$

E. $\frac{9}{\pi}$

24. The region bounded by the graphs of $y = x^2 - 5x + 6$ and $y = 0$ is rotated about the y-axis. The volume of the resulting solid is

A. 10π

B. 52π

C. 5π/6

D. 5π/3

E. 19π/3

25. Let A be the region bounded by $y = \ln x$, the x-axis, and the line $x = e$. Which of the following represents the volume of the solid generated when A is revolved around the y-axis?

A. $\pi \int_0^1 \left(e^2 - e^{2y}\right)dy$

B. $\pi \int_0^1 \left(e^2 - e^{y}\right)dy$

C. $\pi \int_0^1 e^{2y}\,dy$

D. $2\pi \int_1^e \left(e^2 - e^{2y}\right)dy$

E. $\pi \int_0^1 e^2\,dy$

26. Which of the following series diverge?

I. $\sum_{n=1}^{\infty} \frac{4}{3^{n+1}}$

II. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n}$

III. $\sum_{n=3}^{\infty} \frac{3}{n^2+1}$

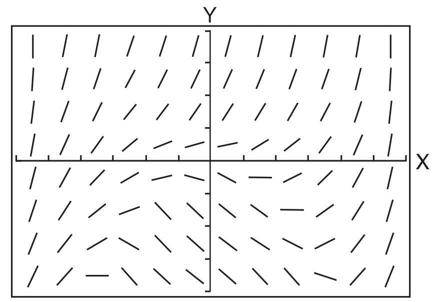
A. none

B. I only

C. II only

D. III only

E. All of them



27. Above is the slope field for which of the following differential equations?

A. $\frac{dy}{dx} = \sin x$

B. $\frac{dy}{dx} = x^2 + y$

C. $\frac{dy}{dx} = 2x + 3$

D. $\frac{dy}{dx} = 3y - 2$

E. $\frac{dy}{dx} = \cos y$

28. A particle moves in the xy-plane so that at any time $t > 0$, $x = \frac{1}{4}t^4 - 3t$ and $y = \frac{1}{3}(3t-5)^4$. The acceleration vector of the particle at $t = 2$ is

A. $\left(-2, \frac{1}{3}\right)$

B. (5, 4)

C. (12, 36)

D. (12, 4)

E. $\left(\frac{-52}{15}, \frac{1}{810}\right)$

Practice Test 2 – BC

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS SECTION ONLY. DO NOT WORK ON ANY OTHER SECTION IN THE TEST.

STOP

Section IB: BC

Time: 50 Minutes

17 Questions

Directions: The 17 questions that follow in Section IB of the exam should be solved using the space available for scratchwork. Select the best of the given choices and fill in the corresponding oval on the answer sheet. Material written in the test booklet will not be graded or awarded credit. Fifty minutes are allowed for Section IB. *A graphing calculator is required for this section of the test.*

Notes: *(1) If the exact numerical value does not appear as one of the five choices, choose the best approximation. (2) For this test, ln* x *denotes the natural logarithm function (that is, logarithm to the base* e*). (3) The domain of all functions is assumed to be the set of real numbers* x *for which* f(x) *is a real number, unless a different domain is specified.*

1. If a, b, and c are constants, what is $\lim_{z \to a}\left(bz^2 + c^2 z\right)$?

- **A.** $a^2b + ac^2$
- **B.** $az^2 + a^2z$
- **C.** $bz^2 + ac^2$
- **D.** $a^2b + c^2z$
- **E.** $bz^2 + c^2z$

2. The slope of the line normal to the curve $g(x) = \sin\frac{x}{2} + 3x$ at the point where $x = \frac{\pi}{7}$ is approximately

- **A.** –0.287
- **B.** 0.449
- **C.** 1.276
- **D.** 2.671
- **E.** 3.487

3. Given $\lim_{x \to 3^+} f(x) = 2$, which of the following MUST be true?

- **A.** $f(3)$ exists
- **B.** $f(x)$ is continuous at $x = 3$
- **C.** $f(3) = 2$
- **D.** $\lim_{x \to 3^-} f(x) = 2$
- **E.** None of these must be true.

4. A point of inflection for the graph of $y = x^3 + 4x^2 - x + 5\sin x$ has x coordinate

- **A.** –4.467
- **B.** –3.273
- **C.** –2.066
- **D.** –1.059
- **E.** –.519

5. If $y = x^{x-2}$, then $\left.\frac{dy}{dx}\right|_{x=3} \approx$

- **A.** 2.358
- **B.** 3.761
- **C.** 4.296
- **D.** 4.553
- **E.** none of these

6. A right circular cylindrical can having a volume of $2\pi \text{ in}^3$ is to be constructed. Find the radius of the can for which the total surface area is a minimum. $\left(V = \pi r^2 h,\ A = 2\pi r^2 + 2\pi rh\right)$

A. 1/4

B. 1/2

C. 1

D. 2

E. 4

7. If $g(x) = \frac{1}{2}|3 - x|$, then the value of the derivative of $g(x)$ at $x = 3$ is

A. $\frac{-1}{2}$

B. $\frac{1}{2}$

C. 0

D. 3

E. nonexistent

8. $\int \frac{dx}{x^2 - x - 2} =$

A. $-\frac{1}{3}\ln\left|\frac{x+1}{x-2}\right| + C$

B. $\ln\left|\frac{x-1}{x+2}\right|^3 + C$

C. $-\frac{1}{x} - \ln|x| - \frac{x}{2} + C$

D. $\frac{1}{3} - \ln\left|\frac{x-1}{x+2}\right| + C$

E. $\ln\left|\frac{x-2}{x+1}\right|^3 + C$

9. For the function $y = x^{100}$ find $\frac{d^{100}y}{dx^{100}}$.

A. 0

B. 100

C. $(100!)x$

D. 100!

E. $100x$

10. $\int_1^2 \sin^5 x\, dx \approx$

A. 0.732

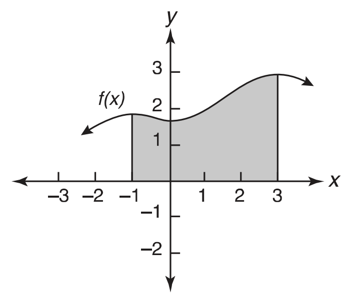
B. 0.815

C. 0.867

D. 0.924

E. 1.173

11. Which of the following is equal to the shaded area in the figure below?



A. $\lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{3}{n}\right) f\left(\frac{3i}{n}\right)$

B. $\lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{4}{n}\right) f\left(-1 + \frac{4i}{n}\right)$

C. $\lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{3}{n}\right) f\left(-1 + \frac{3i}{n}\right)$

D. $\lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{4}{n}\right) f\left(\frac{4i}{n}\right)$

E. none of these

12. What is $\lim_{h \to 0} \frac{e^{x+h} - e^x}{h}$?

A. 0

B. $\ln x$

C. x

D. e^x

E. 1

GO ON TO THE NEXT PAGE

13. The average value of the function $f(x) = x \sin x$ on the closed interval $[1, \pi]$ is approximately

A. 1.326

B. 1.467

C. 2.840

D. 3.142

E. 4.076

14. For what values of x will the series $\sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1)^2}$ converge?

A. $(-1, 1]$

B. $[-1, 1)$

C. $(-1, 1)$

D. $[-1, 1]$

E. $(-\infty, +\infty)$

15. When the value of cos2 is approximated by using the fourth-degree Taylor polynomial about $x = 0$, the value of cos2 is

A. $1 + \frac{4}{2} - \frac{16}{24}$

B. $1 + \frac{4}{2} + \frac{16}{24}$

C. $1 - \frac{4}{2} - \frac{16}{24}$

D. $-1 - \frac{4}{2} + \frac{16}{64}$

E. $1 - \frac{4}{2} + \frac{16}{24}$

16. A curve is described by the parametric equations $x = \frac{1}{4}t^4$ and $y = \frac{1}{3}t^3$. The length of this curve from $t = 0$ to $t = 2$ is given by

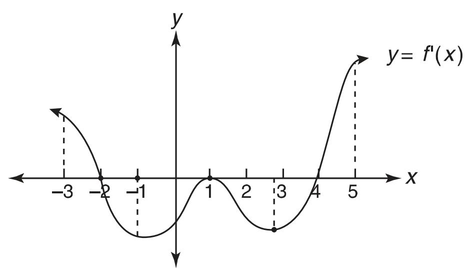
A. $\int_0^2 \sqrt{\frac{t^8}{16} + \frac{t^6}{9}}\, dt$

B. $\int_0^2 \sqrt{\frac{t^4}{4} + \frac{t^3}{3}}\, dt$

C. $\int_0^2 \sqrt{\frac{t^{10}}{400} + \frac{t^8}{144}}\, dt$

D. $\int_0^2 \sqrt{\left(t^3 - t^2\right)^2}\, dt$

E. $\int_0^2 \sqrt{\left(t^6 - t^4\right)}\, dt$



17. Above is the graph of $f'(x)$. On what interval (5) is the graph of $f(x)$ concave upwards?

A. $-3 < x < 1$ and $1 < x < -1$

B. $-2 < x < 1$ and $1 < x < 4$

C. $-1 < x < 3$

D. $-1 < x < 1$ and $3 < x < 5$

E. $-3 < x < 1$

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS SECTION ONLY. DO NOT WORK ON ANY OTHER SECTION IN THE TEST.

STOP

Section II: Free-Response Questions

Section IIA

Time: 45 Minutes

3 Questions

Directions: For the three problems that follow in Section IIA, show your work. Your grade will be determined on the correctness of your method as well as the accuracy of your final answers. Some questions in this section may require the use of a graphing calculator. If you choose to give decimal approximations, your answer should be correct to three decimal places, unless a particular question specifies otherwise. During Section IIB, you will be allowed to return to Section IIA to continue your work on questions 1–3, but you will NOT be allowed the use of a calculator.

Notes: *(1) For this test, ln* x *denotes the natural logarithm function (that is, logarithm to the base* e*). (2) The domain of all functions is assumed to be the set of real numbers* x *for which* f(x) *is a real number, unless a different domain is specified.*

1. Let R be the first quadrant region enclosed by the graphs of $y = \sin x$ and $y = e\left(x^2 - 2\right)$.

(a) Find the area of R.

(b) Set up, but do not evaluate, an integral which represents the length of the boundry of the region R.

(c) The base of a solid is the region R. Cross sections of the solid are semicircles perpendicular to the base and the x-axis, with their diameters on region R. Set up, but do evaluate, an integral which represents the volume of this solid.

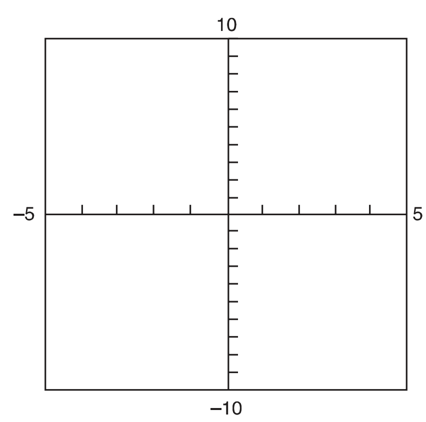
2. Two particles move in the xy-plane. For time $t \geq 0$, the position of particle P is given by $x = t - 3$ and $y = (t - 2)^2$ while the position of particle Q is given by $x = 2t/3 - 5/3$ and $y = 2t/3 + 4/3$.

(a) Find the velocity vector for each particle at $t = 2$.

(b) Set up, but do not evaluate, an integral expression that represents the distance traveled by particle P from $t = 1$ to $t = 4$.

(c) Find the exact time at which the two particles are at the same position at the same time.

(d) In the viewing window at the top of the next page, sketch the paths of particles P and Q, from $t = 0$ until they collide. Indicate the direction of each particle along its path.



VIEWING WINDOW [-5,5] x [-10,10]

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3. Let $f''(x) = 10\sin x - (e^x)\cos x$ for the interval [–2.5, 5].

(a) On what intervals is the graph of $f(x)$ concave down? Justify your answer.

(b) Find the x-coordinates of all relative extrema for the function $f'(x)$. Classify the extrema as relative minima or relative maxima. Justify your answer.

(c) To the nearest tenth, find the x-coordinates of any points of inflection of the graph of $f'(x)$. Justify your answer.

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS SECTION ONLY. DO NOT WORK ON ANY OTHER SECTION IN THE TEST.

STOP

Section IIB

Time: 45 Minutes

3 Questions

Directions: For the three problems that follow in Section IIB, show your work. Your grade will be determined on the correctness of your method as well as the accuracy of your final answers. During Section IIB, you will NOT be allowed the use of a calculator. During this section you will also be allowed to return to questions 1–3 in Section IIA to continue working on those problems, but you will NOT have the use of a calculator.

4. Let $y = f(x)$ be a continuous function such that $\frac{dy}{dx} = 3xy$, $x \geq 0$, and $f(0) = 12$.

(a) Find $f(x)$.

(b) Find $f^{-1}(x)$.

5. Let f be a function that has derivatives of all orders for all numbers. Assume $f(1) = 3$, $f'(1) = -2$, $f''(1) = 7$, and $f'''(1) = -5$

(a) Find the third-degree Taylor polynomial for f about $x = 3$ and use it to approximate $f(3.2)$.

(b) Write the fourth-degree Taylor polynomial for g, where $g(x) = f(x^2 + 3)$ about 3.

(c) Write the third-degree Taylor polynomial for h, where $f(x) = \int_3^x f(t)\,dt$ about $x = 3$.

6. Let f be the function defined as

$$f(x)\begin{cases} 6x + 12 & \text{for } x \leq -2 \\ ax^3 + b & \text{for } -2 < x < 1 \\ 2x + \frac{5}{2} & \text{for } x \geq 1 \end{cases}$$

(a) Find values for a and b such that $f(x)$ is continuous. Use the definition of continuity to justify your answer.

(b) For the values you found in part (a), is $f(x)$ differentiable at $x = -2$? at $x = 1$? Use the definition of the derivative to justify your answer.

IF YOU FINISH BEFORE TIME IS CALLED, CHECK YOUR WORK ON THIS SECTION ONLY. DO NOT WORK ON ANY OTHER SECTION IN THE TEST.

STOP

Practice Test 2 — BC

Answer Key for Practice Test 2 — BC

Section I: Multiple-Choice Questions

Section IA

1. C	**8.** B	**15.** E	**22.** A
2. A	**9.** B	**16.** E	**23.** B
3. D	**10.** C	**17.** D	**24.** C
4. D	**11.** E	**18.** B	**25.** A
5. C	**12.** D	**19.** A	**26.** A
6. C	**13.** C	**20.** A	**27.** B
7. E	**14.** D	**21.** E	**28.** C

Section IB

1. A	**6.** C	**10.** B	**14.** D
2. A	**7.** E	**11.** B	**15.** E
3. E	**8.** A	**12.** D	**16.** E
4. C	**9.** D	**13.** A	**17.** D
5. C			

Unanswered problems are neither right nor wrong, and are not entered into the scoring formula.

Number right = ___________

Number wrong = ___________

Section II: Free-Response Questions

Use the grading rubrics beginning on page 460 to score your free-response answers. Write your scores in the blanks provided on the scoring worksheet.

Practice Test 2 Scoring Worksheet

Section IA and IB: Multiple-Choice

Of the 45 total questions, count only the number correct and the number wrong. Unanswered problems are not entered in the formula.

__________ − (1/4 × __________) = __________

number correct — number wrong — Multiple-Choice Score

Section II: Free-Response

Each of the six questions has a possible score of 9 points. Total all six scores.

Question 1 ______________
Question 2 ______________
Question 3 ______________
Question 4 ______________
Question 5 ______________
Question 6 ______________

TOTAL ______________
Free-Response Score

Composite Score

1.20 × ______________ = ______________
Multiple-Choice Score — Converted Section I Score (do not round)

1.00 × ______________ = ______________
Free-Response Score — Converted Section II Score (do not round)

TOTAL of converted scores = ______________
round to the nearest whole number

Probable AP Grade

Composite Score Range	*AP Grade*
65–108	5
55–64	4
42–54	3
0–41	1 or 2

Please note that the scoring range above is an approximation only. Each year, the chief faculty consultants are responsible for converting the final total raw scores to the 5-point AP scale. Future grading scales may differ markedly from the one listed above.

Answers and Explanations for Practice Test 2 — BC

Section I: Multiple-Choice Questions

Section IA

1. C. Simplify first.

$$g(t) = \frac{e^{\ln t}}{t} = \frac{t}{t} = 1$$

2. A. $xy = 12$

$$\frac{dx}{dt} \cdot x + \frac{dy}{dt} \cdot = 0$$

$$2 \cdot 3 + \frac{dy}{dt} \cdot 4 = 0$$

$$\frac{dy}{dt} \cdot 4 = -6$$

$$\frac{dy}{dt} = \frac{-3}{2}$$

3. D. $\lim\limits_{\theta \to 0}\left(\frac{3\theta}{\tan\theta}\right) = \lim\limits_{\theta \to 0}\left[\left(\frac{\theta}{\sin\theta}\right)(3\cos\theta)\right] = (1)(3)(1) = 3$

or use L'Hôpital's rule:

$$\lim_{\theta \to 0}\left(\frac{3\theta}{\tan\theta}\right) = \lim_{\theta \to 0}\frac{3}{\sec^2\theta} = \frac{3}{1} = 3$$

4. D. $\lim\limits_{x \to \infty} e^{-x} \cdot \ln x$

$= \lim\limits_{x \to \infty} \frac{\ln x}{e^x}$ of the form $\frac{\infty}{\infty}$

using L'Hôpital's rule:

$= \lim\limits_{x \to \infty} \frac{\frac{1}{x}}{e^x}$

$= \frac{0}{\infty}$

$= 0$

5. C. The one-sided limit is approaching 2 from the left, so use the "middle" part of the piece function.

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2-} \left(x^2\right) = 4$

6. C. $x = t^3 - 1 \quad y = t^2$

$\frac{dx}{dt} = 3t^2 \quad \frac{dy}{dt} = 2t$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 2t \cdot \frac{1}{3t^2} = \frac{2}{3t} = \frac{2}{3}t^{-1}$

Then $\frac{d^2y}{dx^2} = \frac{dy^1}{dx} = \frac{dy^1}{dt} \cdot \frac{dt}{dx} = \frac{-2}{3t^2} = \frac{1}{3t^2} = \frac{-2}{9t^4}$

7. E. $\frac{d}{dx}\left[\frac{\cos x}{\sin x + 1}\right] = \frac{(\sin x + 1)(-\sin x) - (\cos x)(\cos x)}{(\sin x + 1)^2}$

$= \frac{-\sin^2 x - \sin x - \cos^2 x}{(\sin x + 1)^2}$

$= \frac{\left(-\sin^2 x + \cos^2 x\right) - \sin x}{(\sin x + 1)^2}$

$= \frac{-1 - \sin x}{(\sin x + 1)^2}$

$= \frac{-1}{1 + \sin x}$

8. B. $y = \sqrt{\ln 3x} = (\ln 3x)^{1/2}$

$\frac{dy}{dx} = \frac{1}{2}(\ln 3x)^{-1/2}\left(\frac{3}{3x}\right)$

$= \frac{1}{2x\sqrt{\ln 3x}}$

9. B. $\int x \sin dx$ Usingintegration by parts,

$= \int u dv$ let $u = x$ and $dv = \sin x$

$= uv - \int v du$ Then $du = dx$ and $v = -\cos x$

$= x(-\cos x) - \int (-\cos x) dx$

$= -x\cos x + \sin x + C$

10. C. $y = \arctan(3x) \Rightarrow \dfrac{dy}{dx} = \dfrac{3}{1+(3x)^2}$

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{3}{1+3^2} = \frac{3}{10}$$

11. E. $\dfrac{d}{dy}\left[\log_3 4y\right] = \dfrac{1}{\ln 3}\cdot\dfrac{4}{4y} = \dfrac{1}{y\ln 3}$

12. D. To find a point of inflection, find the second derivative and do interval testing.

$$y = e^{-x^2} \Rightarrow y' = \left(e^{-x^2}\right)(-2x)$$

$$y'' = \left(e^{-x^2}\right)(-2) + \left(-2x\left[e^{-x^2}(-2x)\right]\right)$$

$$= -2e^{-x^2} + 4x^2 e^{-x^2}$$

$$= e^{-x^2}\left(-2+4x^2\right)$$

$y'' = 0$ or y'' or does not exist

$4x^2 = 2$

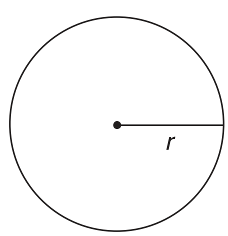
$x^2 = \dfrac{1}{2}$

$x = \dfrac{\pm\sqrt{2}}{2}$

x	$x < \frac{-\sqrt{2}}{2}$	$x = \frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}$	$x = \frac{\sqrt{2}}{2}$	$x > \frac{\sqrt{2}}{2}$
y''	pos	0	neg	0	pos
y	concave up	POI	concave down	POI	concave up

Thus there are points of inflection at $x = \dfrac{\pm\sqrt{2}}{2}$.

13. C. Sketch the circle; r = radius and A = area.



Find dA/dt when $A = 16\pi$, given $dr/dt = 3$.

$A = \pi r^2$

$\dfrac{dA}{dt} = \pi 2r\dfrac{dr}{dt} \quad A = 16\pi \Rightarrow r = 4$

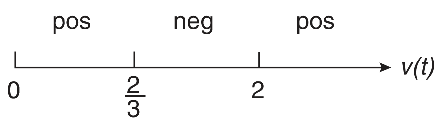
$$\left.\frac{dA}{dt}\right|_{r=4} = 2\pi(4)(3) = 24\pi \text{ m}^2/\text{s}$$

14. D. "moving right" $\Rightarrow v(t) > 0$

$$v(t) = s'(t) \Rightarrow v(t) = 3t^2 - 8t + 4$$

$$v(t) = (3t-2)(t-2)$$

$$v(t) = 0 \Rightarrow t = \frac{3}{2} \text{ or } t = 2$$



Thus $v(t) > 0$ for $0 < t < \frac{2}{3}$ or $t > 2$.

15. E. Use implicit differentiation.

$$xy - x + y = 2$$

$$\left[x\frac{dy}{dx} + y(1)\right] - 1 + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}[x+1] = 1 - y$$

$$\frac{dy}{dx} = \frac{1-y}{x+1}$$

$xy - x + y = 2$ and $x = 0 \Rightarrow y = 2$

Thus (0, 2) is the point of tangency.

$$\left.\frac{dy}{dx}\right|_{x=0} = \frac{1-2}{0+1} = -1 \Rightarrow m_t = -1 \text{ is slope of tangent}$$

Use the point/slope form.

$$y - 2 = (-1)(x-0) \Rightarrow y = -x + 2$$

16. E. Use integration by parts: $\int u\,dv = uv - \int v\,du$

$$u = x \quad dv = e^{2x}$$

$$du = dx \int dv = \int e^{2x}\,dx$$

$$\int dv = \frac{1}{2}\int e^{2x}(2)\,dx$$

$$v = \frac{1}{2}e^{2x}$$

$$\int 3xe^{2x}\,dx = 3\int xe^{2x}\,dx$$

$$= 3\left[x\left(\frac{1}{2}e^{2x}\right) - \int\left(\frac{1}{2}e^{2x}\right)dx\right]$$

$$= \frac{3}{2}xe^{2x} - \frac{3}{2}\int e^{2x}\,dx$$

$$= \frac{3}{2}xe^{2x} - \frac{3}{2}\cdot\frac{1}{2}\int e^{2x}(2)\,dx$$

$$= \frac{3}{2}xe^{2x} - \frac{3}{4}e^{2x} + C$$

17. D. $\int \cot 3\phi \, d\phi = \frac{1}{3} \int \cot 3\phi (3) \, d\phi$

$$= \frac{1}{3} \ln |\sin 3\phi| + C$$

18. B. Use u-substitution. Let $u = \sqrt{x+5}$

$$u^2 = x + 5$$
$$x = u^2 - 5$$
$$dx = 2u\,du$$

$$\int \frac{2x}{\sqrt{x+5}} dx = 2 \int \frac{u^2 - 5}{u} 2u\,du$$
$$= 4 \int (u^2 - 5)\,du$$
$$= 4\left[\frac{u^3}{3} - 5u\right] + C$$
$$= 4\frac{u}{3}\left[u^2 - 15\right] + C$$
$$= \frac{4}{3}\sqrt{x+5}\left[(x+5) - 15\right] + C$$
$$= \frac{4}{3}\sqrt{x+5}(x - 10) + C$$

19. A. The integral $\int_1^2 \frac{dx}{\sqrt[3]{x-1}}$ has a discontinuity at its lower limit of integration, 1.

$$\int_1^2 \frac{dx}{\sqrt[3]{x-1}} = \lim_{c \to 1^+} \int_c^2 (x-1)^{-\frac{1}{3}} dx$$
$$= \lim_{c \to 1^+} \left[\frac{3}{2}(x-1)^{\frac{2}{3}}\right]_c^2$$
$$= \lim_{c \to 1^+} \left[\frac{3}{2}(1) - \frac{3}{2}(c-1)^{\frac{2}{3}}\right]$$
$$= \frac{3}{2} - 0$$
$$= \frac{3}{2}$$

20. A. $\int_0^2 f(x)\,dx = \int_0^1 2x \, dx + \int_1^2 (3x^2 - 1)\,dx$

$$= \left[x^2\right]_0^1 + \left[x^3 - x\right]_1^2$$
$$= [1 - 0] + \left[(8 - 2) - (1 - 1)\right]$$
$$= 1 + 6 = 7$$

21. E. area $= \int_1^e \frac{x^3+2}{x}\,dx$

$$= \int_1^e \left(x^2 + \frac{2}{x}\right)dx$$

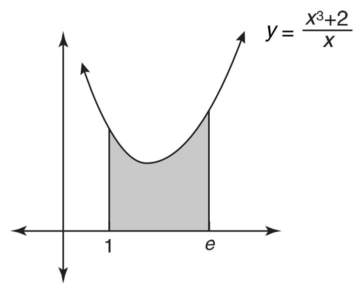
$$= \left[\frac{x^3}{3} + 2\ln x\right]_1^e$$

$$= \frac{e^3}{3} + 2\ln(e) - \left(\frac{1}{3} + 2\ln(1)\right)$$

$$= \frac{e^3}{3} + 2 - \frac{1}{3} - 0$$

$$= \frac{e^3}{3} + \frac{5}{3}$$

$$= \frac{e^3+5}{3}$$



22. A. The phrase "rate of change proportional to amount present" translates into

$$\frac{dy}{dt} = ky$$

Solving this differential equation yields

$y = Ce^{kt}$

(For this work, see the differential equations section on page 292.)

"54 grams present initially" $y = 54$ when $t = 0$

$54 = Ce^{\circ} \Rightarrow C = 54$

Thus $y = 54e^{kt}$

"42 grams after 120 years" $\Rightarrow y = 42$ when $t = 120$

$$42 = 54e^{120k}$$

$$\frac{7}{9} = e^{120k}$$

$$\ln\frac{7}{9} = 120k$$

$$k = \frac{1}{120}\ln\frac{7}{9}$$

Thus $y = 54e^{kt}, k = \frac{1}{120}\ln\frac{7}{9}$

23. B. average value $= \frac{1}{b-a}\int_a^b f(x)\,dx$

$$= \frac{1}{\frac{\pi}{3} - 0}\int_0^{\pi/3} \tan x\,dx$$

$$= \frac{3}{\pi}\left[-\ln|\cos x|\right]_0^{\pi/3}$$

$$= \frac{-3}{\pi}\left[\ln\frac{1}{2} - \ln 1\right]$$

$$= \frac{-3}{\pi}\ln\frac{1}{2}$$

$$= \frac{-3}{\pi}\ln 2^{-1}$$

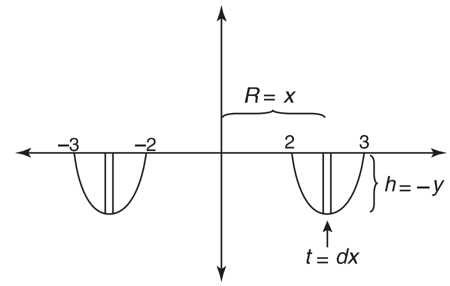
$$= \frac{3}{\pi}\ln 2$$

24. C. $y = x^2 - 5x + 6$

$0 = (x-2)(x-3)$

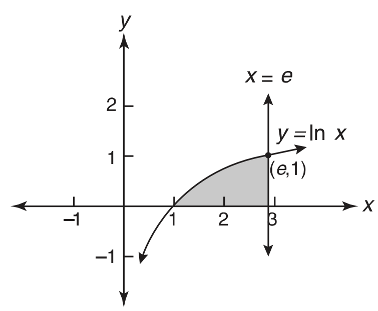
$x = 2\ x = 3$

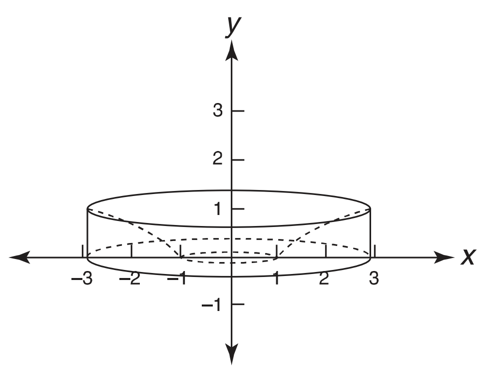
$V_{\text{shell}} = 2\pi Rht$



$$V = 2\pi \int_2^3 x(-y)\,dx$$
$$= 2\pi \int_2^3 x(-x^2 + 5x - 6)\,dx$$
$$= 2\pi \int_2^3 x(-x^3 + 5x^2 - 6x)\,dx$$
$$= 2\pi \left[\frac{-x^4}{4} + \frac{5x^3}{3} - 3x^2\right]_2^3$$
$$= 2\pi \left[\left(\frac{-81}{4} + 45 - 27\right) - \left(-4 + \frac{40}{3} - 12\right)\right]$$
$$= \frac{5\pi}{6}$$

25. A. Sketch the area and solid as shown.





By washers, vertical axis ⇒ *dy*.

Area *A* extends from $y = 0$ to $y = 1$ along the y-axis ⇒ $\int_0^1$.

Washers: $\pi \int_a^b \left[\begin{pmatrix} \text{outer} \\ \text{radius} \end{pmatrix}^2 - \begin{pmatrix} \text{inner} \\ \text{radius} \end{pmatrix}^2 \right] dy \Rightarrow$

$V = \pi \int_0^1 \left(e^2 - e^{2y}\right) dy$

26. A.

I. $\sum_{n=1}^{\infty} \frac{4}{3^{n+1}}$

$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right|$

$= \lim_{n \to \infty} \left| \frac{4}{3^{n+2}} \cdot \frac{3^{n+1}}{4} \right|$

$= \lim_{n \to \infty} \left| \frac{1}{3} \right| = \frac{1}{3} < 1$

So series converges by the Ratio Test.

II. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$= \sum_{n=2}^{\infty} (-1)^n \cdot \frac{1}{n}$

with $u_n = 1/\text{n}$,

i) $u_{n>0}$

ii) $u_{\text{n}+1} < u_n$

and

iii) $\lim_{n \to \infty} u_n = 0$

So series converges by the Alternating Series Test.

III. $\sum_{n=3}^{\infty} \frac{3}{n^2 + 1}$

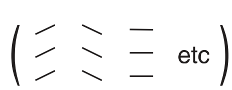
$= 3 \sum_{n=3}^{\infty} \frac{1}{n^2 + 1}$

Compare this series to the p-series. $\sum_{n=3}^{\infty} \frac{1}{n^2}$, which is convergent ($p > 1$). Since $\frac{1}{n^2+1} < \frac{1}{n^2}$, given series also converges by the Comparison Test.

$\Rightarrow$ So none of the series diverge.

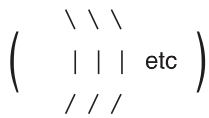
27. B. In general for slope fields:

i) if the columns of slopes are the same



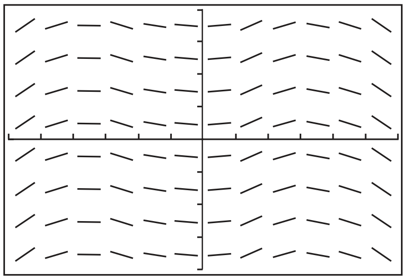
then the differential equation depends on just x — as in A and C

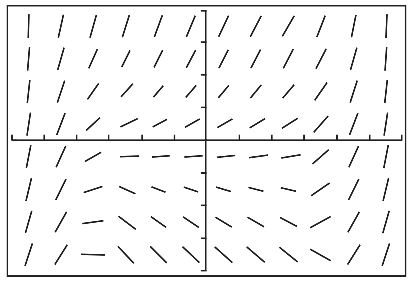
ii) if the rows of slopes are the same

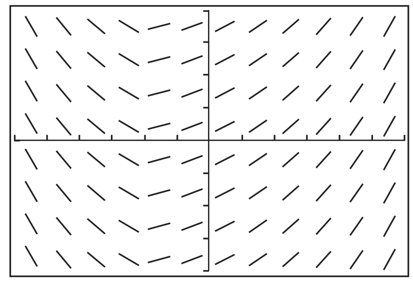


then the differential equation depends on just y — as in D and E

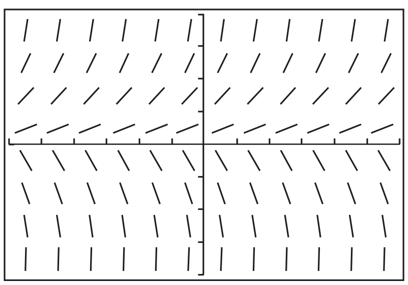
iii) otherwise, depends on x and y — as in Graph B

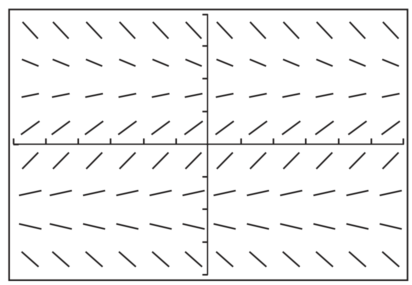






A $\rightarrow \dfrac{dy}{dx} = \sin x$ B $\rightarrow x^2 + y$ C $\rightarrow \dfrac{dy}{dx} = 2x + 3$





D $\rightarrow \dfrac{dy}{dx} = 3y - 2$ E $\rightarrow \dfrac{dy}{dx} = \cos y$

28. C.

$$x(t) = \frac{1}{4}t^4 - 3t \qquad y(t) = \frac{1}{3}(3t-5)^4$$

$$x'(t) = t^3 - 3 \qquad y'(t) = \frac{4}{3}(3t-5)^3 \cdot 3$$

$$x''(t) = 3t^2 \qquad = 4(3t-5)^3$$

$$x''(2) = 3 \cdot 2^2 \qquad y''(t) = 12(3t-5)^2 \cdot 3$$

$$x''(2) = 12 \qquad = 36(3t-5)^2$$

$$y''(2) = 36(6-5)^2$$

$$y''(2) = 36$$

Since acceleration vector is ($x''(2)$, $y''(2)$) desired vector is (12, 36).

Section IB

1. **A.** The limit is to be taken as z approaches a. Because a, b, and c are constants, the argument is just a polynomial function with z as the independent variable, so substitute a into the function for z.

   $\lim\limits_{z \to a}\left(bz^2 + c^2 z\right) = b(a)^2 + c^2(a) = a^2 b + ac^2$

2. **A.** The slope of a normal line is the negative reciprocal of the slope of the tangent line. Find $-1/g'(\pi/7)$ with your calculator. Let $y_1 = \sin(x/2) + 3x$

```
nDeriv(Y1,X,π/7)
                3.487463936
-1/Ans
               -.2867413164
```

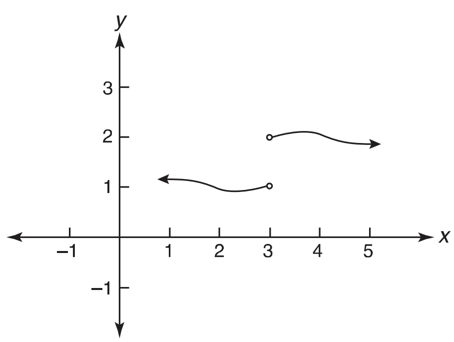
   or find $g'(x)$

   first: $g(x) = \sin\frac{x}{2} + 3x$

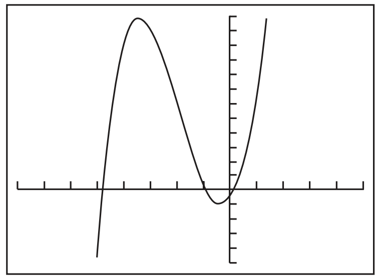
   $g'(x) = \frac{1}{2}\cos\frac{x}{2} + 3$

   and now find $\frac{-1}{g'\left(\frac{\pi}{7}\right)}$.

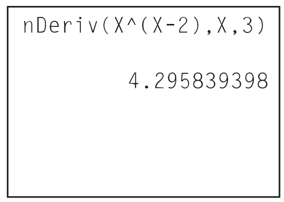
3. **E.** By finding one (or more) counterexamples, it is possible to eliminate each of choices A through D in turn. The following graph shows such a counterexample. The stated condition, $\lim\limits_{x \to 3^+} f(x) = 2$, is true, whereas choices A through D are not true.



4. **C.** The graph is shown on the next page. Note that choices A and D approximate zeros of the function; B a relative maximum, and E a relative minimum. Choice C –2.066 best approximates where the graph changes its concavity — in this case from concave downward to concave upward.



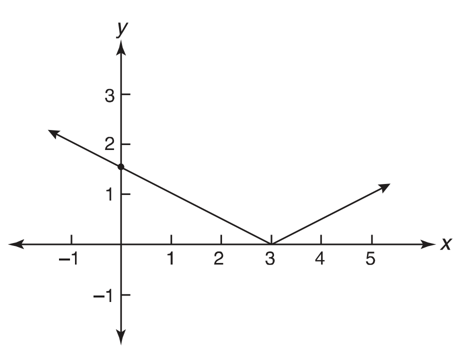
5. C. Use a calculator to find the derivative at a point.



6. C.

$$V=\pi r^2 h$$
$$2\pi=\pi r^2 h$$
$$\frac{2}{r^2}=h$$
$$A=2\pi r^2+2\pi rh$$
$$A(r)=2\pi r^2+2\pi r\left(\frac{2}{r^2}\right)$$
$$=2\pi r^2+\frac{4\pi}{r}$$
$$=2\pi r^2+4\pi r^{-1}$$
$$A'(r)=4\pi r-\frac{4\pi}{r^2}$$
$$0=4\pi r-\frac{4\pi}{r^2}$$
$$0=4\pi r^3-4\pi$$
$$0=4\pi\left(r^3-1\right)$$
$$r=1$$

7. E. A quick sketch of the graph (with or without a calculator) shows a sharp turn at $x=3$, so the derivative at that point is nonexistent.



8. A. Using integration by "partial fractions," first separate the integrand into its component parts.

$\frac{1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} \Rightarrow$ Note: $x^2 - x - 2 = (x + 1)(x - 2)$ multiplying by $(x + 1)(x - 2)$ gives:

$1 = A(x - 2) + B(x + 1)$

when $x = -1$ $\quad 1 = A(-3) + B(0)$

$\left\langle -\frac{1}{3} = A \right\rangle$

when $x = 2$ $\quad 1 = A(0) + B(3)$

$\left\langle \frac{1}{3} = B \right\rangle$

Then $\int \frac{dx}{(x+1)(x-2)}$

$$= \int \left(\frac{-\frac{1}{3}}{x+1} + \frac{\frac{1}{3}}{x-2} \right) dx$$

$$= -\frac{1}{3}\ln|x+1| + \frac{1}{3}\ln|x-2| + C$$

$$= -\frac{1}{3}\left(\ln|x+1| - \ln|x-2|\right) + C$$

$$= -\frac{1}{3}\ln\left|\frac{x+1}{x-2}\right| + C$$

9. D. Find the first few derivatives and look for a pattern.

$$y = x^{100}$$

$$\frac{dy}{dx} = 100x^{99}$$

$$\frac{d^2y}{dx^2} = 100 \cdot 99x^{98}$$

$$\frac{d^3y}{dx^3} = 100 \cdot 99 \cdot 98x^{97}$$

$$\frac{d^4y}{dx4} = 100 \cdot 99 \cdot 98 \cdot 97x^{96}$$

$\vdots$

$$\frac{d^{100}y}{dx^{100}} = 100 \cdot 99 \cdot 98 \cdot 97 \cdots 3 \cdot 2 \cdot\cdot 1x^0 = 100$$

10. B. Find the value of the definite integral with your calculator.

```
Y1=(sin X)^5

fnInt(Y1,X,1,2)
        .814956842
```

11. B. The shaded area is equivalent to the definite integral.

$$\int_{-1}^{3} f(x)\,dx$$

The base of the area is 4 units, and if this base were divided into n equal subintervals, each subinterval would have a width of $4/n$. Thus the right-hand endpoint of the ith rectangle is given by $-1 + 4/n$.

Thus $\int_{-1}^{3} f(x)\,dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left(\frac{4}{n}\right) f\left(-1 + \frac{4i}{n}\right)$

12. D. The question is in the form of the definition of the derivative, using h instead of Δx where $f(x) = e^x$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = e^x \Rightarrow f'(x) = e^x$$

13. A. By the definition of average value, use a calculator to find

$$\frac{1}{\pi - 1} \int_{1}^{\pi} x \sin x\,dx$$

```
fnInt(Xsin X,X,1,π)
        2.840423975
Ans/(π-1)
        1.326313839
```

14. D. Using Ratio Test: $\sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1)^2}$

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \to \infty} \left| \frac{x^{n+2}}{(n+2)^2} \cdot \frac{(n+1)^2}{x^{n+1}} \right| = \lim_{n \to \infty} \left| x \cdot \frac{(n+1)^2}{(n+2)^2} \right|$$

$$= |x| \cdot \lim_{n \to \infty} \left| \frac{(n+1)^2}{(n+2)^2} \right| = |x| \cdot 1 = |x|$$

This ratio will be less than 1 when $|x| < 1 \Rightarrow -1 < x < 1$

Testing the endpoints:

i) when $x = -1$: series becomes $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$ since $0 < |u_{n+1}| \le |u_n|$ and $\lim_{n \to \infty} u_n = 0$, alternating series test indicates series is convergent $\Rightarrow$ include $x = -1$

ii) when $x = 1$: series becomes $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ since $\frac{1}{(n+1)^2} < \frac{1}{n^2}$ is a convergent p series, comparison test indicates series is convergent $\Rightarrow$ include $x = 1$ $\therefore$ interval of convergence is $-1 \le x \le 1$ or $[-1,1]$

15. E.

$$f(x) = \cos x \qquad f(0) = 1$$
$$f'(x) = -\sin x \qquad f'(0) = 0$$
$$f''(x) = -\cos x \qquad f''(0) = -1$$
$$f'''(x) = \sin x \qquad f'''(0) = 0$$
$$f^{iv}(x) = \cos x \qquad f^{iv}(0) = 1$$
$$T_3 f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{iv}(0)x^4}{4!}$$
$$= 1 + 0 - \frac{1x^2}{2} + 0 + \frac{1x^4}{24}$$
$$= 1 - \frac{x^2}{2} + \frac{x^4}{24}$$
$$f(2) \approx 1 - \frac{2^2}{2} + \frac{2^4}{24} = 1 - \frac{4}{2} + \frac{16}{24}$$

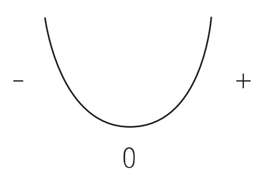
16. E. For a curve described by parametric equations, arc length = $\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$$x = \frac{1}{4}t^4 \qquad y = \frac{1}{3}t^3$$
$$\frac{dx}{dt} = t^3 \qquad \frac{dy}{dt} = t^2$$

so arc length $= \int_0^2 \sqrt{\left(t^3\right)^2 + \left(t^2\right)^2}\, dt$

$$= \int_0^2 \sqrt{t^6 + t^4}\, dt$$

17. D. For a function $f(x)$, its graph is concave upward if its slope is increasing on an interval.

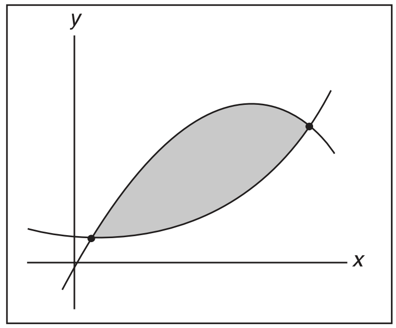


For graph of some $f(x)$, slope changes from negative (–), to zero (0), to positive (+) when the graph is concave upward. On the given graph of $f'(x)$, slope is increasing (in this case, y-coordinate is changing from negative to zero to positive) on the intervals $-1 < x < 1$ and $3 < x < 5$.

Section II: Free-Response Questions

Section IIA

1. (a) (see graph below)



$\sin x = e^{x^2-2}$

Using calculator to find (x), x = 0.138394 or x = 1.40959

Let a = .0138394, let b = 1.30959

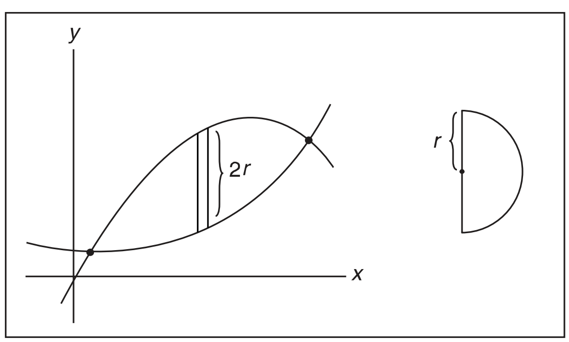
area = $\int_a^b \left(\sin x - e^{x^2-2}\right)dx = .401$

(b) boundary length = L

$L = \int_a^b \sqrt{1+(\cos x)^2}\,dx + \int_a^b \sqrt{1+\left(\left(e^{x^2-2}\right)2x\right)^2}\,dx$

For y = $f(x)$ from $x = a$ to $x = b$, arc length is $\int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx$

(c)



$r = \left(\sin x - e^{x^2-2}\right)/2$

area of cross section $= \left(\pi r^2\right)/2 = (\pi/8)\left(\sin x - e^{x^2-2}\right)^2$

so volume $= (\pi/8)\int_a^b = \left(\sin x - e^{x^2-2}\right)^2\,dx$

Grading Rubric

(a) 3 points 2: correct integral

1: limits of integration

1: correct integral

1: answer

(b) 3 points 1: correct derivatives of sinx and e^{x^2-2}

1: correct arc length integral

1: correct limits of integration and sum of integrals

(c) 3 points 1: correct radius

1: correct integral

1: correct constant (π/8)

2. (a) $V_P = (dx/dt, dy/dt) = (1, 2t - 4)$ at $t = 2$, $V_P = (1, 0)$

$V_Q = (2/3, 2/3)$ at $t = 2$, $V_Q = (2/3, 2/3)$

(b) $\text{dist} = \int_1^4 \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dx$

$= \int_1^4 \sqrt{1^2 + (2t-4)^2}\, dx$

(c) $X_P = X_Q$

$t - 3 = 2t/3 - 5/3$

$3t - 9 = 2t - 5$

$t = 4$

at $t = 4$ both P and Q are at points having x-coordinate of 1

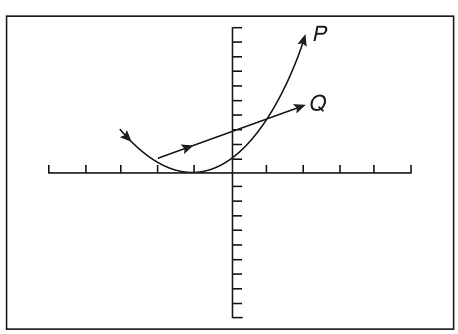
$Y_1 = (t-2)^2$ $\quad$ $Y_2 = 2t/3 + 4/3$

$t = 4, Y_1 = (4-2)2$ $\quad$ $t = 4\, Y_2 = 8/3 + 4/3$

$Y_1 = 4$ $\quad$ $Y_2 = 4$

at $t = 4$ particles P and Q both occupy the point (1, 4)

(d) see graph below



Grading Rubric

(a) 2 points 1: correct derivatives

1: correct vectors

(b) 2 points 1 correct integral

1: limits of integration

(c) 3 points 1: sets $X_P = X_Q$, or $Y_P = Y_Q$

1: correct solution for t

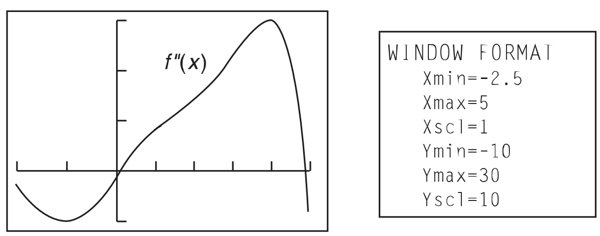
1: other coordinate shown to be equal

(d) 2 points 1: graph and direction for P

1: graph and direction for Q

3. (a) $f(x)$ is concave down when $f''(x)$ is negative

The graph of $f''(x)$ is shown below from the calculator with a window of X[–2.5, 5] and Y[–10, 30].



The zeros of $f''(x)$ are $x = 0.11$ and $x = 4.61$, and from the graph, $f''(x) < 0$ when $-2.5 < x < 0.11$ and $4.61 < x < 5$, so these are the intervals where $f(x)$ is concave down.

(b) $f'(x)$ will have relative extrema when its derivative, $f''(x)$, changes sign. From the graph in part (a), this occurs when $x = 0.11$ and $x = 4.61$. At $x = 0.11$, $f'(x)$ has a relative minimum, since $f''(x)$ changes from negative to positive. At $x = 4.61$, $f'(x)$ has a relative maximum, since $f''(x)$ changes from positive to negative.

(c) By definition, a function has points of inflection where concavity changes and the tangent line exists. A function is concave up when its derivative is increasing; a function is concave down when its derivative is decreasing. So, $f'(x)$ will have points of inflection where its derivative, $f''(x)$, changes from increasing to decreasing or decreasing to increasing, that is, where $f''(x)$ has relative extrema. From the graph in part (a), $f''(x)$ has relative extrema when $x = -1.6$ and $x = 3.8$, thus these are the inflection points of $f'(x)$.

Grading Rubric

(a) 3 points 1: indicates need for $f''(x)$ negative

1: finds zeros of $f''(x)$ via calculator

1: correct intervals

(b) 4 points 2: correct x-coordinates

2: justification: sign change in $f''(x)$

(c) 2 points 1: correct x-coordinates

1: justification: relative extrema of $f''(x)$

4. (a) Solve the differential equation by separating the variables.

$$\frac{dy}{dx} = 3xy$$

$$\frac{1}{y}dy = 3xdx$$

$$\int \frac{1}{y}dy = \int 3xdx$$

$$\ln y = \frac{3}{2}x^2 + C_1$$

$$e^{(3/2)x^2 + C_1} = y$$

$$e^{(3/2)x^2} e^{C_1} = y$$

$$C_2 e^{(3/2)x^2} = y$$

$$f(0) = 12 \Rightarrow x = 0 \text{ when } y = 12$$

$$C_2 e^{(3/2)(0)} = 12 \Rightarrow C_2 = 12$$

Thus $f(x) = 12e^{(3/2)x^2}$

(b) To find the inverse, interchange x and y.

$$12e^{(3/2)x^2} = y$$

$$12e^{(3/2)y^2} = x$$

$$e^{(3/2)y^2} = \frac{x}{12}$$

$$\frac{3}{2}y^2 = \ln \frac{x}{12}$$

$$y = \pm\sqrt{\frac{2}{3}\ln \frac{x}{12}}$$

But y must be greater than or equal to 0, so

$$f^{-1}(x) = \sqrt{\frac{2}{3}\ln \frac{x}{12}}$$

Grading Rubric

(a) 6 points
- 1: separates the variables
- 2: integrates correctly
- 2: finds correct C
- 1: expresses $f(x)$

(b) 3 points
- 1: interchanges x and y on $f(x)$
- 1: solves for y
- 1: expresses $f^{-1}(x)$

5. (a) $P_3(f)(x) = f(1) + \frac{f'(1)(x-3)}{1!} + \frac{f''(1)(x-3)^2}{2!} + \frac{f'''(1)(x-3)^3}{3!}$

$$= 3 - 2(x-3) + (7/2)(x-3)^2 - (5/6)(x-3)^3$$

$$f(3.2) \approx 3 - 2(0.2) + (7/2)(0.2)^2 - (5/6)(0.2)^3$$

$$\approx 2.733$$

(b) $P_4(g)(x) = P_2(f)(x^2+3)$

$$= 3 - 2\left((x^2+3)-3\right) + 3.5\left((x^2+3)-3\right)^2$$

$$= 3 - 2x^2 + 3.5(x^4)$$

(c) $P3\ (h)(x) = \int_3^x \left(3 - 2(t-3) + (7/2)(t-3)^2\right) dt$

$$= \left[3t - (t-3)^2 + (7/6)(t-3)^3\right]_3^x$$

$$= 3x - (x-3)^2 + (7/6)(x-3)^3 - \left[3(3) - 0 + 0\right]$$

$$= 3(x-3) - (x-3)^2 + (7/6)(x-3)^3$$

Grading Rubric

(a) 4 points 3: correct $P_3(f)(x)$ <–1> for each error

1: correct approximation for $f(3.2)$

(b) 2 points 2: correct $P_4(g)(x)$ <–1> for each error

(c) 3 points 1: correct set up

1: correct antiderivatives

1: correct answer

6. (a) For $f(x)$ to be continuous, $y = ax^3 + b$ must contain the points (–2, 0) and (1, 9/2).

$$\left.\begin{array}{l} 0 = -8a + b \\ \frac{9}{2} = a + b \end{array}\right\} \Rightarrow \begin{array}{l} a = \frac{1}{2} \text{ and } b = 4 \\ \text{will guarantee continuity} \end{array}$$

$$f(x) = \begin{cases} 6x + 12 & \text{for } x \le -2 \\ \frac{1}{2}x^3 + 4 & \text{for } -2 < x < 1 \\ 2x + \frac{5}{2} & \text{for } x \ge 1 \end{cases}$$

Justification:

$$f(x) \text{ is continuous at } x = a \text{ if } \begin{cases} f(a) \text{ exists} \\ \lim_{x \to a} f(x) \text{ exists} \\ \lim_{x \to a} f(x) = f(a) \end{cases}$$

At $x = -2$: $f(-2) = 0$

$$\left.\begin{aligned}\lim_{x \to -2^-} f(x) &= \lim_{x \to -2^-} (6x + 12) = 0\\ \lim_{x \to -2^+} f(x) &= \lim_{x \to -2^+} \left(\frac{1}{2}x^3 + 4\right) = \frac{1}{2}(-8) + 4 = 0\end{aligned}\right\}$$

$$\Rightarrow \lim_{x \to -2^-} f(x) = 0 = f(-2)$$

Thus $f(x)$ is continuous at $x = -2$.

At $x = 1$: $f(1) = \frac{9}{2}$

$$\left.\begin{aligned}\lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} \left(\frac{1}{2}x^3 + 4\right) = \frac{9}{2}\\ \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} \left(2x + \frac{5}{2}\right) = \frac{9}{2}\end{aligned}\right\} \Rightarrow \lim_{x \to 1} f(x) = \frac{9}{2} = f(1)$$

Thus $f(x)$ is continuous at $x = 1$.

(b) $$f(x)\begin{cases} 6x + 12 & \text{for } x \le -2 \\ \frac{1}{2}x^3 + 4 & \text{for } -2 < x < 1 \\ 2x + \frac{5}{2} & \text{for } x \ge 1 \end{cases}$$

$$f'(x)\begin{cases} 6 & \text{for } x \le -2 \\ \frac{3}{2}x^2 & \text{for } -2 < x < 1 \\ 2 & \text{for } x \ge 1 \end{cases}$$

$$\left.\begin{aligned}f'(-2^-) &= 6\\ f'(-2^+) &= 6\end{aligned}\right\} \Rightarrow \text{differentiable at } x = -2$$

Justification:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$$f'(-2^-) = \lim_{x \to 2^-} \frac{(6x + 12) - 0}{x - (-2)} = \lim_{x \to 2^-} \frac{6(x + 2)}{x + 2} = 6$$

$$f'(-2^+) = \lim_{x \to 2^+} \frac{\left(\frac{1}{2}x^3 + 4\right) - 0}{x - (-2)} = \lim_{x \to 2^+} \frac{\frac{1}{2}(x^3 + 8)}{x + 2}$$

$$= \lim_{x \to -2^+} \frac{\frac{1}{2}(x + 2) - (x^2 - 2x + 4)}{x + 2} = \frac{1}{2}(12) = 6$$

Thus $f(x)$ is differentiable at $x - 2$.

$$\left.\begin{aligned}f'(1^-) &= \frac{3}{2}\\ f'(1^+) &= 2\end{aligned}\right\} \Rightarrow f(x)\,x \text{ is not differentiable at } x = 1$$

Justification:

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

$$f'(1^-)=\lim_{x\to 1^-}\frac{\left(\frac{1}{2}x^3+4\right)-\frac{9}{2}}{x-1}=\lim_{x\to 1^-}\frac{\frac{1}{2}\left(x^3-1\right)}{x-1}$$

$$=\lim_{x\to 1^-}\frac{\frac{1}{2}(x-1)\left(x^2+x+1\right)}{x-1}=\frac{3}{2}$$

$$f'(1^+)=\lim_{x\to 1^+}\frac{\left(2x+\frac{5}{2}\right)-\frac{9}{2}}{x-1}=\lim_{x\to 1^+}\frac{2(x-1)}{x-1}=2$$

Thus $f(x)$ is not differentiable at $x=1$.

Grading Rubric

(a) 3 points
- 1: finds values of a and b
- 2: justifies with the definition of continuity

(b) 6 points
- 1: determines differentiability at $x=-2$
- 2: justifies with definition of derivative, including one-sided limits
- 1: determines differentiability at $x=1$
- 2: justifies with definition of derivative, including one-sided limits