§P.3—Simplifying Expressions

Jaime Escalante, the most famous high school calculus teacher of all time, had a banner in his classroom that read "Calculus doesn't have to be made easy, it already is." How true that message is. As I already mentioned, AP Calculus is simply two ideas, instantaneous rate of change, which is a division process, and areas of irregular regions, a multiplicative process. To make it even simpler, then, calculus is about dividing and multiplying. The thing that makes calculus such a challenging course is ironically not the calculus itself, but rather all the tedious algebra that is needed to implement the calculus.

In this section, we will examine some of the algebraic "gymnastics" we'll need.

A mathematical expression is one of two types, either a phrase like "when in doubt, multiply by the page number" or a collection of numbers, variables, and/or operational signs. Simplifying varieties of the second type allow us to work with smaller, more manageable things without a loss of generality. Working with expressions is very different that working with conditional equations. There are only a handful of things you can do with an expression without changing its value. These include factoring, adding clever forms of zero (completing the square), and multiplying by a clever form of one (getting a common denominator).

What is meant by "simplifying" is to make an expression smaller, more condensed, contain less "stuff", with fewer terms or fewer factors, etc. There's no definitive "simplest form" in general, but I think you will begin to notice when an expression is stripped down as far it possibly can. On multiple-choice questions, of course, you might have to simplify to a point that resembles (exactly) the correct answer choice.

We'll start with simplifying by factoring, of which there are several things to look for, summarized in the chart below

1. Look for a common factor
2. Look for a special product like
   - Difference of two squares: $a^2 - b^2 = (a-b)(a+b)$
   - Perfect square trinomial: $a^2 \pm 2ab + b^2 = (a \pm b)^2$
   - Sum/Difference of Cubes
     - $a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$
     - $a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right)$
3. Factorable trinomial (Target Sum/Target Product)
4. Guess and Check
5. Grouping
6. Using synthetic division

Let's work a few.

Example 1:

Simplify $\dfrac{x^3-9x}{x^2-7x+12}$.

That was too easy.

Example 2:

Simplify $\dfrac{4x^3-8x^2-20x+40}{2x-4}$.

Already the level of sophistication is increasing, but hopefully you're still following along okay. Hopefully you're not too bored either.

A complex (or compound) fraction is an expression that has fractions within fractions. It might look intimidating, but I assure you it's all bark and no bite. Unlike it's close relative, the compound fracture, dealing with these fractions is neither painful nor excruciatingly painful. There is a tidy algebraic maneuver that will efficiently eradicate these complex or compound fractions. It involves multiplying by a clever form of one that involves the least common multiple (LCM) of all the "miniature" denominators.

Here's one.

Example 3:

Simplify $\dfrac{\frac{1}{x}-\frac{1}{5}}{\frac{1}{x^2}-\frac{1}{25}}$.

Sometimes these complex fractions can be in disguise. It's your job to recognize them. This will require a reminder about negative and rational exponents. Here it is:

$$a^{\frac{m}{n}}=\sqrt[n]{a^m}=\left(\sqrt[n]{a}\right)^m \text{ and } b\cdot a^{-m}=\frac{b}{a^m}$$

This means that all radical expressions can be written as rational exponents, with the root off the radical being the denominator of the exponent. Factors with negative exponents can become a factor on the other "side" of a fraction simply by changing the sign of its exponent.

Expressions like $\dfrac{1}{2\sqrt[3]{x^2}}$ can be equivalently written as $\dfrac{1}{2}x^{\frac{2}{3}}$.

When simplifying expressions involving radicals or variables in the denominator, it is easier to write them in exponential form; however, when evaluating these expressions for particular x-values, it is easier to do so in their radical or denominator form.

Example 4:

Simplify $\dfrac{3-6(x-2)^{-1}}{6-12x\left(x^2-4\right)^{-1}}$.

Sometimes fractional expressions contain radicals or rational exponents (hidden radicals, remember?). Simplifying these type of expressions usually involves a process called rationalization, which is a fancy name for algebraically moving radicals from either numerator to denominator or vice-versa. You did this quite a bit in Precalculus, especially in your Trig unit. Ratios like $\dfrac{1}{\sqrt{2}}$ became the equivalent $\dfrac{\sqrt{2}}{2}$. The purpose of this was to make the Unit Circle more uniform. In general, both expressions are considered equally simplified. In fact, one could make an argument that the first one is more simplified. There will be times, though, when working with variable expressions where we will need to pull out the same trick to circumvent a algebraic tight spot.

Example 5:

Rationalize $\dfrac{\sqrt{x+3}-\sqrt{3}}{x}$.

Sometimes the process involves only numbers.

Example 6:

Simplify $\frac{4}{1-\sqrt{5}}$ by rationalizing.

Sometimes simplifying expressions means creating them first, making a mountain out of a mole hill before making the mountain back into a mole hill.

Before doing that, it's important to understand function notation and how to evaluate a function.

Function notation $y = f(x)$ is so useful because it provides an efficient way to see both input and output, independent and dependent variable, *x*- and *y*- value. Your experience in evaluating functions is probably limited to plugging in specific values in for *x*. If you've studied composition of functions, they you've been lucky enough to plug variable expression containing *x* in for *x*. When this happens, it's very useful to revert back to your days before algebra, when you used spaces, rectangular boxes, or parenthesis to represent an unknown, as in the follow conditional equation from those glorious days of your mathematical yesteryear.

$$5+\boxed{}=8$$

Back then you got a gold star for writing a "3" inside the box. You perhaps then got a warm, fuzzy feeling on the inside when your math teacher told you that you just solved and algebraic equation (or not). We'll do the same thing evaluating the function $f(x)=\frac{2x^2+x}{\sqrt{x+1}}$.

Rewrite it as $f(\quad)=\frac{2(\quad)^2+(\quad)}{\sqrt{(\quad)+1}}$

Now whatever appears in the parenthesis on the left side of the equation will appear in each and every parenthesis in the expression on the right-hand side of the equation.

We'll evaluate this function now for selected inputs.

Example 7:

If $f(\quad) = \dfrac{2(\quad)^2 + (\quad)}{\sqrt{(\quad)+1}}$, evaluate the following.

(a) $f(1) =$

(b) $f(-1) =$

(c) $f(x+2) =$

(d) $f\left(e^x\right) =$

(e) $f(\sin(x-4)) =$

(f) $f(\text{pink elephant}) =$

You get the idea. While this "template writing" is not a necessary step for success in AP Calculus, it provides a systematic way to avoid careless errors and perhaps a world of confusion. Remember that it's the algebra that makes calculus so difficult. If there is a proven method for making it less difficult, it's worth implementing. Believe me, one missed negative sign can ruin your whole day. It's very frustrating to have to retrace your steps to find a subtle error. It's often better to just rework the problem a second time working more carefully, more slowly, and of course, more correctly.

Whenever we plug in a variable expression for x into a function, we are creating a new function, much like we did with transformations. This process is a type of **composition**. Understanding composition of functions will be critical to your success later on when you learn to antidifferentiate and integrate.

Example 8:

If $f(x) = 3x^2 - 5x + 1$, simplify the expression $\dfrac{f(x+h) - f(x)}{h}$.

Almost more important than composition of functions is the **decomposition** of functions. This involves identifying which function is on the "inside" and which function containing it is on the "outside" of a given function.

Example 9:

Decompose $h(x)=3\sqrt{4x-1}$ into two functions *f* and *g* such that $h(x)=(f\circ g)(x)=f(g(x))$.

Very often you will have to simplify expressions or equations involving exponential expressions and/or logarithmic equations. The rules for simplifying either type of expressions are very similar, since and exponent is nothing more than a log, and a log is nothing more than an exponent. Here are the basic properties you'll be working with.

<u>**Exponents**</u>	<u>**Logarithms**</u>
$a^0=1,\ a\neq 0$	$\ln 1=0$
$a^1=a$	$\ln e=1$
$a^m\cdot a^n=a^{m+n}$	$\ln mn=\ln m+\ln n$
$\dfrac{a^m}{a^n}=a^{m-n}$	$\ln\dfrac{m}{n}=\ln m-\ln n$
$\left(a^m\right)^n=a^{mn}$	$\ln m^n=n\ln m$
$a^{-m}=\dfrac{1}{a^m},\ a\neq 0$	$e^{\ln x}=x=\ln e^x$
$a^{\frac{m}{n}}=\sqrt[n]{a^m}=\left(\sqrt[n]{a}\right)^m$	$\log_b x=\dfrac{\ln x}{\ln a}$

- Logs were invented by Scottish mathematician John Napier. Originally called "Artificial Numbers," logs are very real indeed," and Napier's logs gave us the rules for working strictly with the exponents of numbers written as powers of a common base. This was useful for working with very small, microscopic numbers as well as very large, astronomical numbers. Scientific notation is based on this using base ten, the common base.
- Any log expression can be equivalently be written as and exponential expression. The conversion formula is

$$\log_b x=y$$
$$\Leftrightarrow$$
$$b^y=x$$

In calculus, we will primarily use base e, rather than base 10, the common base. The number *e*, not to be confused with the letter "e" or television network *E!*, is approximately $2.718\ldots$. It was discovered and named by Swiss mathematician Leonhard Euler (pronounced "oiler"). It is arguably the most famous and important irrational number in all of mathematics. Called the **natural base**, this number is not only healthier and lower in low-density lipoproteins than base 10, but occurs often in nature. If you have an aversion to calling it the natural base, feel free to call it by some of its common (er, natural) monikers: Euler's number, the Banker's number, or $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$.

Example 10:

Find the exact solution to the equation $(6)\left(3^{2x}\right) = 2^{x+1}$, then give a 3-decimal approximation. Verify by solving the equation graphically on your calculator.

Logs make evaluating and simplifying expressions so much fun and sometimes frustrating. Because there are so many different forms of the same correct answer, you need to be adept and agile enough to easily manipulate between different equivalent forms of the same expression. The AP exam doesn't require you to simplify your answers on the free-response portion, but on the multiple-choice section, you might have to simplify your answer beyond a point you would normally stop. Here's and example that made it on the Silver Screen in the 1988 movie "Stand and Deliver" starring Edward James Olmos as the late Calculus guru Jaime Escalante. It was an actual question on the 1985 AP Calculus exam (non-calculator portion).

Which of the following is equal to $\ln 4$?

(A) $\ln 3 + \ln 1$ (B) $\dfrac{\ln 8}{\ln 2}$ (C) $\int_1^4 e^t\,dt$ (D) $\int_1^4 \ln x\,dx$ (E) $\int_1^4 \frac{1}{t}\,dt$

You don't know how to "translate" answer choices (C), (D), and (E) yet, but by the end of the year, once you've mastered how to do that, this question will read more like the following:

Which of the following is equal to $\ln 4$?

(A) $\ln 3 + \ln 1$ (B) $\dfrac{\ln 8}{\ln 2}$ (C) $e^4 - e^1$ (D) $4\ln 4 - 3$ (E) $\ln 4 - \ln 1$

The first two answer choices are bait for the easy prey who haven't memorized or sufficiently practiced their log properties. Choices (C) and (E) should be noticeably wrong for anyone with log or "*e*" experience, including starting campfires and winning spelling bees. By process of elimination, the correct answer choice must be (E). This can be arrived at by two different methods.

$$\begin{array}{l} \ln 4 - \ln 1 \\ \ln 4 - 0 \\ 4 \end{array} \qquad \text{or} \qquad \begin{array}{l} \ln 4 - \ln 1 \\ \ln\left(\dfrac{4}{1}\right) \\ \ln 4 \end{array}$$

Perhaps a ***better*** question would be the following:

Example 11:

Which of the following is NOT equal to $\ln 4$?

(A) $\ln 4 - \ln 1$ (B) $2\ln 2$ (C) $\dfrac{\ln 16}{2}$ (D) $\ln 8 - \ln 2$ (E) $e^{\ln(\ln 4)}$ (F) $\ln e^{\ln 4}$ (G) $\dfrac{\ln 4}{\ln 1}$

Note: On the actual AP exam, the last answer choice is NOT always the correct answer choice. ***Be careful of overgeneralizations and unproven theories based on limited empirical evidence.***