§1.2—Properties of Limits

When working with limits, you should become adroit and adept at using limits of generic functions to find new limits of new functions created from combinations and modifications to those generic functions. I'll show you what I mean, but first, some important properties of limits that make it all work.

Properties of Limits

If $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} g(x)$ also exists too, as well, where k is a constant, then the following are true:

1. (addition/subtraction) $\lim_{x\to c}\left[f(x)\pm g(x)\right]=\lim_{x\to c} f(x)\pm \lim_{x\to c} g(x)$

2. (constant multiple) $\lim_{x\to c} k\cdot f(x)=k\lim_{x\to c} f(x)$

3. (multiplication) $\lim_{x\to c}\left[f(x)\cdot g(x)\right]=\lim_{x\to c} f(x)\cdot \lim_{x\to c} g(x)$

4. (division) $\lim_{x\to c}\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, $\lim_{x\to c} g(x)\neq 0$

5. (exponentiation) $\lim_{x\to c}\left[f(x)\right]^n=\left[\lim_{x\to c} f(x)\right]^n$

6. (composition) If $\lim_{x\to c} g(x)=L$, then $\lim_{x\to c} f(g(x))=\lim_{x\to L} f(x)$

In general, the limit can be taken of each piece including a variable in any expression independently, then, the results of these limits may be combined using the algebraic rules of the expression, for example:

$$\lim_{x\to c}\frac{\sqrt{3x+f(x)}}{x\cdot g^2(x)}=\frac{\sqrt{3\lim_{x\to c} x+\lim_{x\to c} f(x)}}{\lim_{x\to c} x\cdot\left(\lim_{x\to c} g(x)\right)^2}$$

*NOTE: These functions need not be continuous, so be careful concluding something such as $\lim_{x\to c} f(x)=f(c)$ or that just because one of the limits is DNE that the entire limit is too (especially for Property 6)!!!! When a function is not continuous, the limit of the composite function may still exist, it just must be examined at each step from both sides carefully!! WHAT STARTS AS A TWO-SIDED LIMIT MAY, IN FACT, BECOME A ONE-SIDED LIMIT!!!!!

Example 1:

Given $\lim\limits_{x\to 3} f(x)=8$, $\lim\limits_{x\to 3} g(x)=-2$, and $\lim\limits_{x\to 3} h(x)=0$, find the limits that exist.

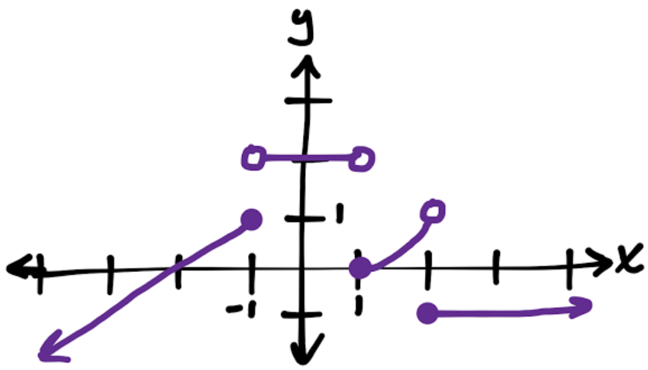
(a) $\lim\limits_{x\to 3}\left[2f(x)-4g(x)\right]=$

(b) $\lim\limits_{x\to 3}\left[2g(x)\right]^2=$

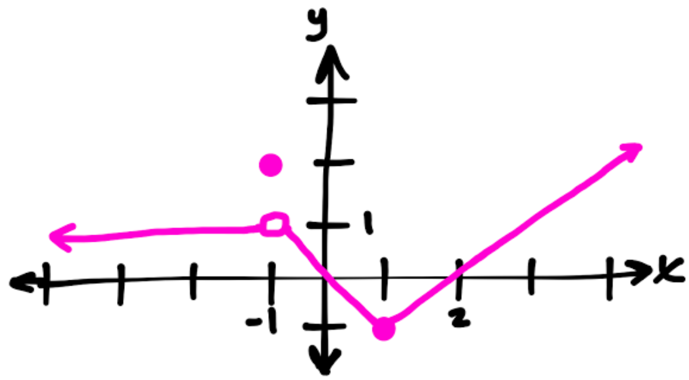
(c) $\lim\limits_{x\to 3}\left(\dfrac{\sqrt[3]{f(x)}}{g(x)}+\dfrac{4h(x)}{x+7}\right)=$

Example 2:

Given the graphs of f and g are given below.



graph of f



graph of g

Determine whether the following limits exist. If they do, then find the limit.

(a) $\lim\limits_{x\to 0}\left[2[f(x)]^2+3g(x)\right]=$

(b) $\lim\limits_{x\to -3}\dfrac{x^2 g(x)}{f(x)}=$

(c) $\lim\limits_{x\to -3} g(f(x))=$

(d) $\lim\limits_{x\to -1} g\left(x^2\right)=$

(e) $\lim\limits_{x\to 1^-}\dfrac{2f(x)}{[g(x)]^2}=$

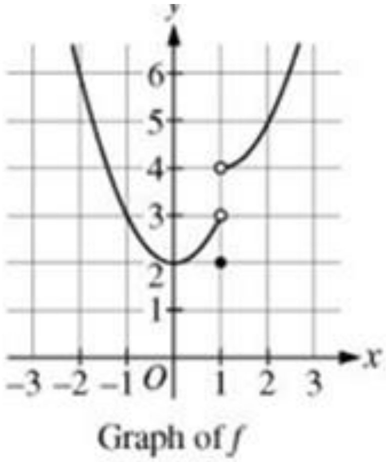
(f) $\lim\limits_{x\to -1^+}\left(5-\sqrt{x^2(3+f(x))}\right)=$

(g) $\lim\limits_{x\to 2} f(x)\cdot g(x)=$

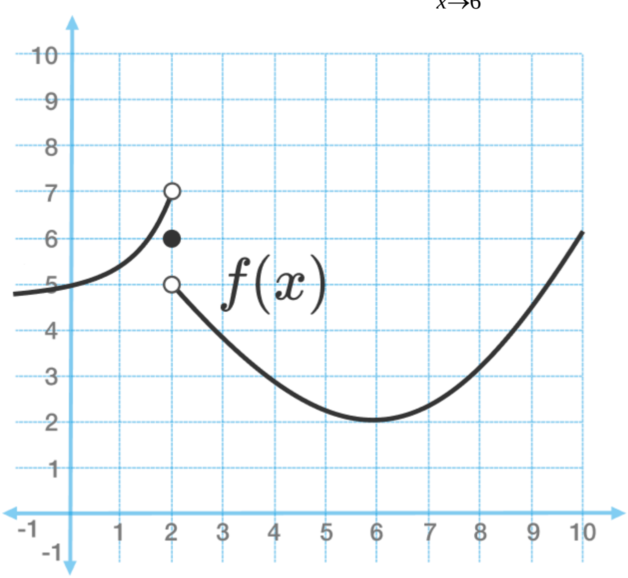
Example 3:

Let's explore some limits for which we must carefully heed the warning in the above NOTE!

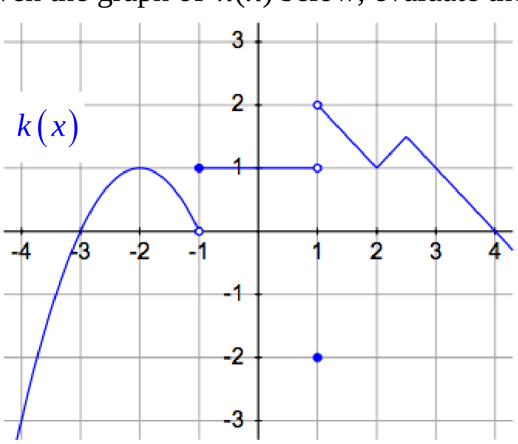
(a) (BC 2016 IE, #88) Given the graph below, evaluate $\lim_{x\to 0} f\left(1-x^2\right)$.



(b) Given the graph below, evaluate $\lim_{x\to 6} f\left(f(x)\right)$.



(c) Given the graph of $k(x)$ below, evaluate the following: **YOU TRY!!**



(i) $\lim_{x\to 3} k\left(k(x)\right)$

(ii) $\lim_{x\to 2} k\left(k(x)\right)$

(iii) $\lim_{x\to -1^-} k\left(k(x)\right)$

(iv) $\lim_{x\to 0} k\left(\cos x\right)$

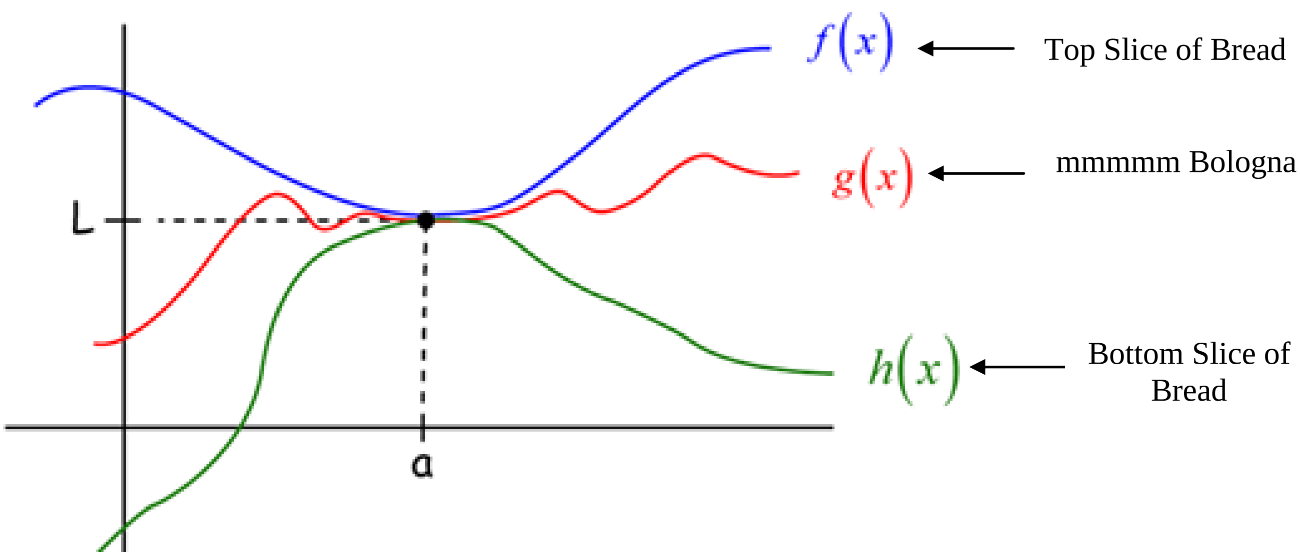
Fun, right!! All that analyzing makes me hungry!!! Sandwich, anyone?

The Squeeze (or SANDWICH) Theorem

If $h(x) \le g(x) \le f(x)$ for all x, except possibly at $x = a$, and **if** $\lim_{x \to a} h(x) = L = \lim_{x \to a} f(x)$,

then $\lim_{x \to a} g(x) = L$

If you have a function that is everywhere contained between two other functions, then at the point where the two outer functions are sandwiched or squeezed together through a single point, then the one in between them **must** pass through that point as well. Think of it as thousands of concert goers from all over the stadium leaving the concert at the end of the night through a **single** turnstile.



The theorem itself is easy enough to understand. It's the recognizing it then formulating your argument that require a little training.

Example 3:

If $-3x \le g(x) \le x^2 + x + 3$, find $\lim_{x \to -1} g(x)$.

Summary

1. Calculate the limit of the top and bottom piece of bread separately.
2. If they are the same, restate or state the squeeze compound inequality.
3. Say, "so, by the Squeeze Theorem...," then state the limit of the unknown sandwiched function.
4. Smile and eat a sandwich (optional).