§4.2—Definite Integrals & Numeric Integration

Calculus answers two very important questions. The first, how to find the instantaneous rate of change, we answered with our study of the derivative. We are now ready to answer the second question: how to find the area of irregular regions.

We start by introducing sigma notation.

The sum S of n terms $a_1,\ a_2,\ a_{3,\ \ldots,}\ a_n$ is written as

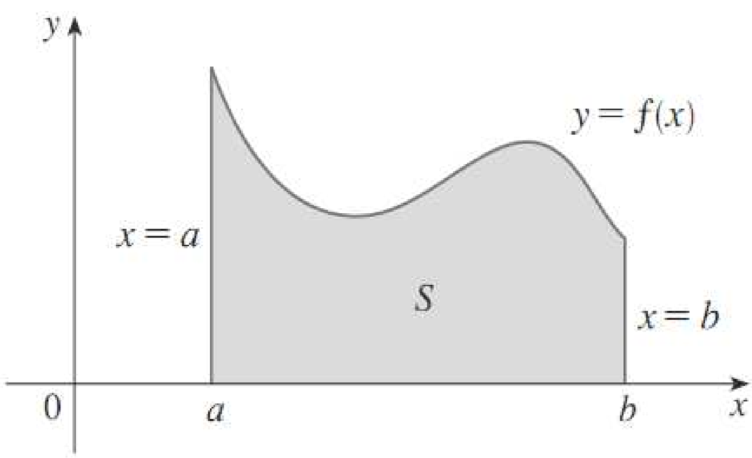
$$S=\sum_{i=1}^{n}a_i=a_1+a_2+a_3+\cdots+a_n$$

where i is called the **index of summation**, a_i is the ***i*th term** of the sum, and the **lower and upper bounds** of the summation are 1 and n.

Example 1:

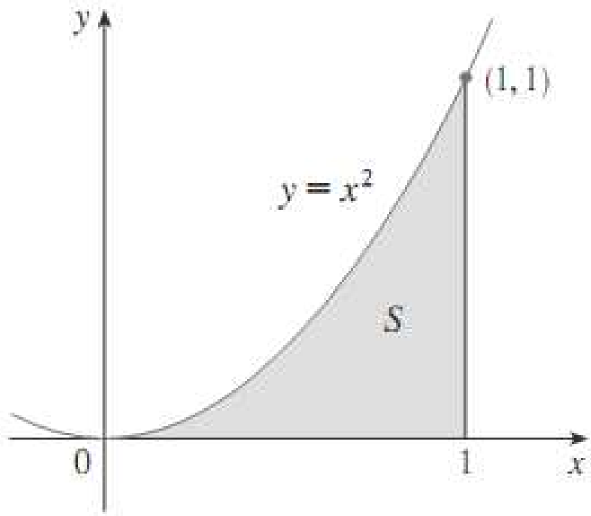
Evaluate a) $\sum_{i=1}^{3}i^2$ b) $\sum_{n=0}^{5}\frac{x^{n+1}}{2^n}$

We will now approximate an irregular area bounded by a function, the x-axis between the vertical lines $x=a$ and $x=b$, like the one below, by finding the areas of many rectangles and summing them up.



Example 2:

Use 4 subintervals of equal width to approximate the area under the parabola $f(x) = x^2$ from $x = 0$ to $x = 1$, notated as region *S*, using the indicated method. Compare to the actual area using your calculator's numeric integration capabilities.



(a) Rectangles using the left-endpoint, L_4

(b) Rectangles using the right endpoint, R_4

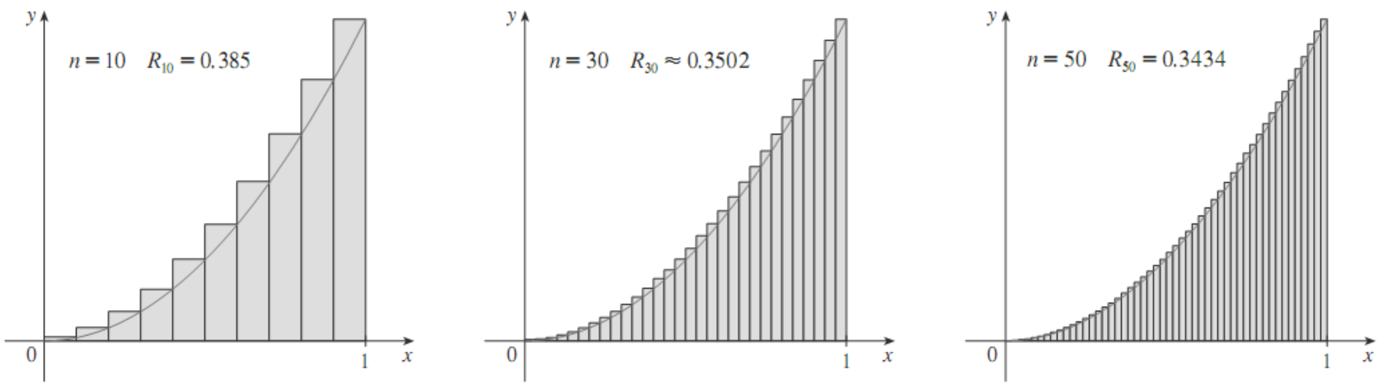
(c) Rectangles using the midpoint, M_4

(d) Trapezoids, T_4

(e) Your calculator's numeric integration capability.

In this case, finding the area approximation using the left-endpoints of the intervals, L_4, gave us an under-approximation for the actual area. Using the right-endpoints, R_4, gives us an over-approximation. Together, these give us an upper and lower bound for the actual area (Note: depending on whether the function is increasing or decreasing, L_n or R_n could either be an upper or lower bound.)

If we desire better approximations of the area, we could partition our area into smaller subintervals using more rectangles. The following chart shows the areas of the same region *S*, using *n* rectangles of equal width using both the left-endpoint and right-endpoint methods.



n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3282500	0.3383500
1000	0.3328335	0.3338335

One can see the limiting process in action from the chart above. As *n* approaches infinity, the area approximations approach the actual area, each converging on the true value of the area.

The process of finding the sum of the areas of rectangles to approximate area of a region is called **Riemann Sums**, after Bernhard Riemann, who pioneered the process.

Riemann proved that the finite process of adding up rectangular areas could be found by a routine analytic process know as **definite integration**. Here's the essence of his great, time-saving work. For $f(x) \geq 0$, on the interval $[a,b]$

$$Area = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i \cdot \Delta x\right)\Delta x = \int_a^b f(x)\,dx$$

Example 3:

Approximate the definite integral $\int_1^9 \sqrt{x}\, dx$ using 3 subintervals of equal width using each of the following methods. Determine if each approximation is an over or an under approximation:

(a) Left Riemann Sums

(b) Right Riemann Sums

(c) Trapezoids

Sometimes we can use known geometric formulas to come up with ACTUAL values of integrals rather than simply approximations.

Example 4:

Evaluate $\int_0^4 (2x)\, dx$ by expressing the definite integral geometrically.

All of the functions/graphs we've dealt with here so far have been nonnegative on the intervals we were interested in. This is not always the case.

Area Under the Curve:

If $y = f(x)$ is **nonnegative** and integrable over a closed interval $[a,b]$, then the **area under the curve** $y = f(x)$ **from *a* to *b*** is the definite integral of *f* from *a* to *b*.

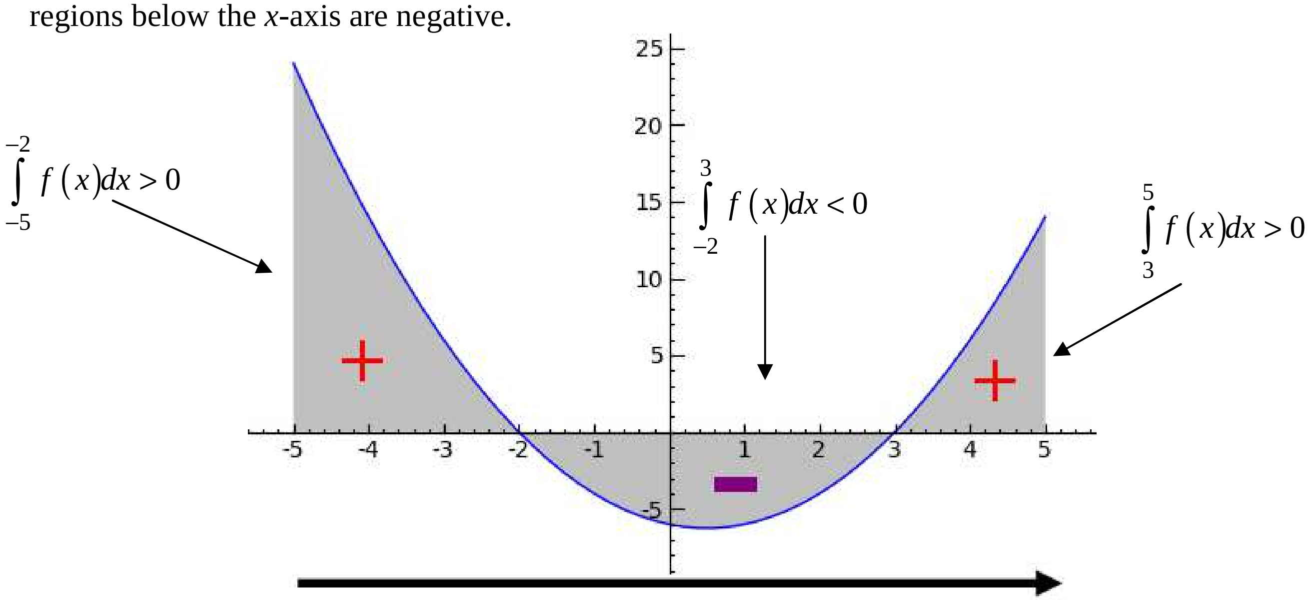
$$A = \int_a^b f(x)\,dx$$

If $y = f(x)$ is **negative** and integrable over a closed interval $[a,b]$, then the **area under the curve** $y = f(x)$ **from *a* to *b*** is the **OPPOSITE** of the definite integral of *f* from *a* to *b*.

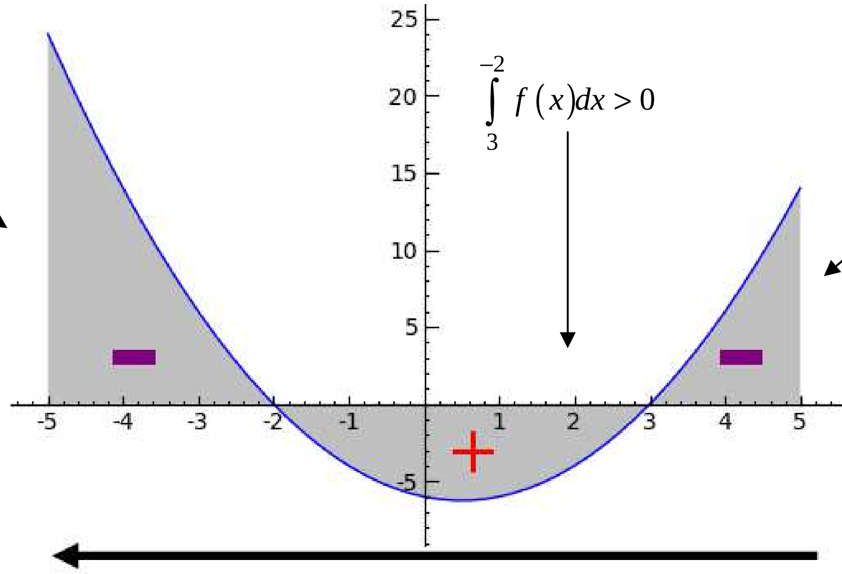
$$A = -\int_a^b f(x)\,dx$$

In General, $\int_a^b f(x)\,dx$ does **NOT** give us the area but rather the **NET** accumulation over the interval from $x = a$ to $x = b$. If $y = f(x)$ is both positive and negative on closed interval $[a,b]$, then $\int_a^b f(x)\,dx$ will NOT give us the area. In this case, the definite integral is a vector—direction matters as much as magnitude.

- When integrating from **left to right** (chronological order), regions above the *x*-axis are positive and regions below the *x*-axis are negative.



- When integrating from **right to left**, regions above the *x*-axis are negative and regions below the *x*-axis are positive.



This second result can be summarized, in general, this way:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

When using the definite integral to help you find area, you cannot always simply evaluate $\int_a^b f(x)\,dx$ if you are integrating over a interval containing negative *y*-values. It is important to remember the following:

AREA IS ALWAYS POSITIVE! AREA IS ALWAYS POSITIVE! AREA IS ALWAYS POSITIVE!

If we are integrating by hand, we must decide if and where the graph crosses the *x*-axis, then split up our interval, manually making negative regions positive. The following property will help accomplish this:

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Example 5: USING INTEGRALS TO FIND AREAS

If $f(x) = \begin{cases} 3, & x < -1 \\ 2 - x, & x \geq -1 \end{cases}$, write and evaluate an integral expression that gives the area of the region bounded by the graph of $f(x)$ and the *x*-axis on the interval $-3 \leq x \leq 3$.

There is another way to find the area of a region, provided you are permitted to use your calculator . . .

Theorem: Area of a region on a calculator

If $f(x)$ is a function defined on an interval $[a,b]$, the area of the region, *A*, bounded by $f(x)$ and the *x*-axis is given by

$$A = \int_a^b |f(x)|\,dx$$

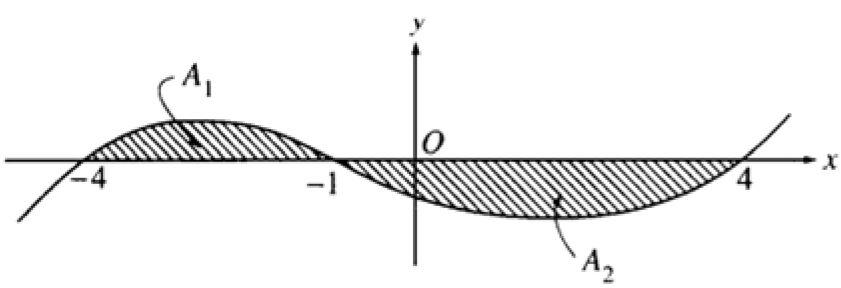
Example 6:

Use your calculator to find the area of the region bounded by the graph of $f(x) = e^x - 3$, the *x*-axis, and the vertical lines $x = 0$ and $x = 3$. Sketch and identify the region first.

So, we can use definite integrals to help us find areas, but we can also use areas to help us find definite integrals. REMEMBER, DEFINITE INTEGRALS DON'T ALWAYS GIVE AREA, BUT RATHER NET ACCUMULATION. Just as before, we are responsible for assigning the regions the correct sign (positive or negative), depending on whether we are **integrating from left to right or right to left** and whether the ***y*-values are positive or negative**.

Example 7: USING AREAS TO FIND INTEGRALS

The graph of $y = f(x)$ is shown below. If A_1 and A_2 are positive numbers that represent the areas of the shaded regions, then find, in terms of A_1 and A_2, the following:



(a) $\int_{-4}^{-1} f(x)\,dx$

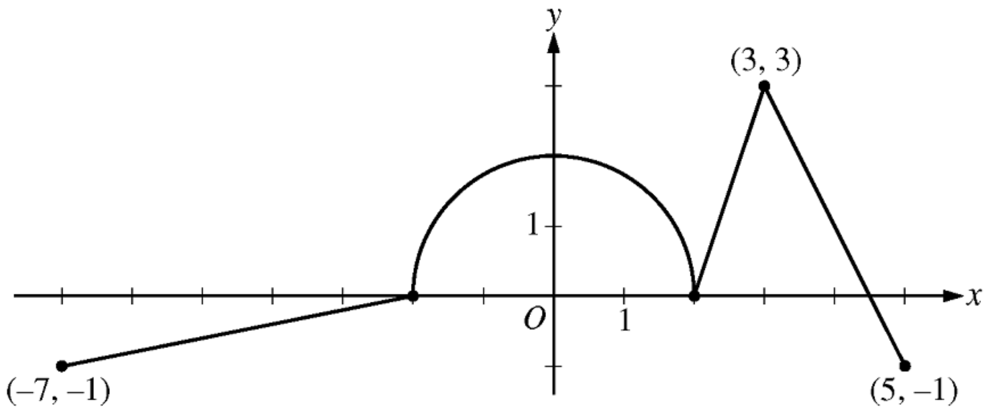
(b) $\int_{-1}^{4} f(x)\,dx$

(c) $\int_{4}^{-1} f(x)\,dx$

(d) $\int_{-4}^{4} f(x)\,dx$

(e) $\int_{-4}^{4} f(x)\,dx - 2\int_{-1}^{4} f(x)\,dx$

Example 8:



The graph above is of a function $y = f(x)$. It is composed of three line segments and a semicircle.

(a) Find the equation of the line segment on the interval $3 \le x \le 5$, and use it to find the *x*-intercept of the line segment.

(b) Evaluate $\int_{-7}^{0} f(x)\,dx$

(c) Evaluate $\int_{5}^{2} f(x)\,dx$

All the functions/graphs we have seen so far have been continuous functions, and each has had a definite integral value. We say these functions are **integrable**.

Theorem: Continuity Implies Integrability

All continuous functions are integrable. That is, if a function *f* is continuous on a the closed interval $[a,b]$, then *f* is integrable on $[a,b]$.

$$C \to I$$
$$\neg I \to \neg C$$

Notice this theorem does NOT say that non-continuous functions aren't integrable.

Example 9:

Evaluate $\int_{-2}^{3} \frac{|x|}{x}dx$ by graphing the function $f(x) = \frac{|x|}{x}$ and using areas.

Functions are not always given to us in equation or graphical form. We can find areas when our function is given to us in either data form as well

Example 10:

x		0	0.5	1	1.5	2	2.5	3
$f(x)$		2	4	6	7	4	1	5

$f(x)$ is a continuous function such that $f(x) \geq 0$ for all x. Selected values are given in the table above.

(a) Approximate $\int_0^3 f(x)\,dx$ using numeric methods as indicated by the data.

(b) Could any of these integral approximations represent approximations of the area of a region?

(c) Approximate $f'(1)$.

Example 11:

x		0	1	3	6	6.6	8	10
$f(x)$		4	3	3	1	5	8	10

If $f(x)$ is a continuous function for all x, given selected values of f above,

(a) Approximate $\int_1^8 f(x)\,dx$ using numeric methods (reread the definite integral).

(b) Could any of these approximations represent the approximate areas of a region?

(c) Approximate $f'(7)$.

Here is a summary of some important integral properties:

Properties of integrals:

1. If f is defined at $x=a$, then $\int_a^a f(x)\,dx=0$
2. If f is integrable on $[a,b]$, then $\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx$
3. $\int_a^b c\,dx=c(b-a)$
4. $\int_a^b [f(x)\pm g(x)]\,dx=\int_a^b f(x)\,dx\pm\int_a^b g(x)\,dx$
5. $\int_a^b cf(x)\,dx=c\int_a^b f(x)\,dx$
6. $\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx$

Example 12:

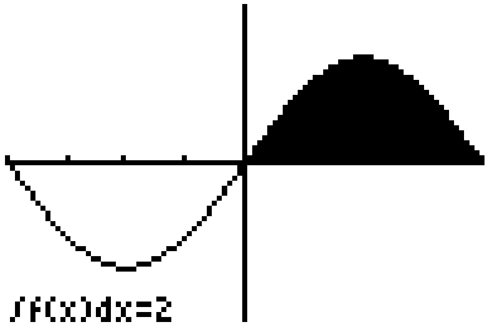
If $\int_0^{10} f(x)\,dx=17$ and $\int_0^{8} f(x)\,dx=12$, find $\int_8^{10}(3f(x)+2)\,dx$

Example 13:

If $\int_4^{-6} f(x)\,dx=-3$, $\int_3^{7} f(x)\,dx=9$, and $\int_4^{7} f(x)\,dx=5$, find $\int_{-6}^{3}\left(\frac{f(x)}{2}-3\right)dx$

Example 14:

If $\int_0^{\pi} \sin x dx = 2$, use this fact and the symmetry of the graph of $f(x) = \sin x$ to find the following:



(a) $\int_{\pi}^{2\pi} \sin x dx$ (b) $\int_0^{2\pi} \sin x dx$ (c) $\int_{\pi}^{\pi/2} \sin x dx$ (d) $\int_0^{\pi} (2 + \sin x) dx$ (e) $\int_0^{\pi} 2\sin x dx$

(f) $\int_2^{\pi+2} \sin(x-2) dx$ (g) $\int_0^{2\pi} \sin\left(\frac{x}{2}\right) dx$ (h) $\int_{-\pi}^{\pi} \sin x dx$ (i) $\int_0^{\pi} \cos x dx$ (j) $\int_0^{2\pi} \cos x dx$