§4.4—Integration by *u*-sub & pattern recognition

Example 1:

Evaluate $\frac{d}{dx}\left[\tan\left(e^{4x^2}\right)\right]=$

Example 2:

Evaluate $\int 8x\cdot e^{4x^2}\cdot\sec^2\left(e^{4x^2}\right)dx=$

We can think of composite functions as being a single function that, like a big box can, contains other functions inside of it (smaller boxes inside the big box). When differentiating such functions, we are "unpacking" these boxes, creating multiple factors in our derivative. When going in reverse, since there is no product or quotient rule for integration, we must recognize the factors in our integrand as coming from "unpacking" the original outer function/big box.

When integrating by pattern recognition, our job is to perform the chain rule in reverse, "packing" back up all the smaller factors (boxes) back into the single big factor (box). This also makes it easier to move . . . onto the next problem.

In general, if

$$\frac{d}{dx}\left[f\left(g\left(x\right)\right)\right]=f'\left(g\left(x\right)\right)\cdot g'\left(x\right)$$

If we solve this differential equation, take the integral of both sides, we get

$$f\left(g\left(x\right)\right)+C=\int f'\left(g\left(x\right)\right)\cdot g'\left(x\right)dx$$

We state this formally as a theorem for integrating a composite function.

Antidifferentiation of a composite function

Let g be a function whose range is an interval, I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f\left(g\left(x\right)\right)\cdot g'\left(x\right)dx = F\left(g\left(x\right)\right)+C$$

In the integrand above, the "inside" function, $g\left(x\right)$, the function whose derivative is the other factor(s) is sometimes called u, where the "outside" function containing u and the function determining the rule of integration, can be called $f\left(u\right)$. In this case we will use differential form and call $u'\ \ du$.

If we rewrite the original integral in terms of u and du, we call it u-substitution.

Rewriting the above rule, we get
Let $u = g\left(x\right)$ and $du = g'\left(x\right)dx$

$$\int f\left(u\right)du = F\left(u\right)+C$$

This formal rewriting of our original integral in terms of a new, temporary, "dummy" variable u can be helpful, but is not always necessary, nor desirable. If you are able to identify both the "inner" and "outer" functions, with a little practice, you will be able to integrate quickly in your head, as the rule of integration will only involve the "outer" function, with a possible "correction" of a scalar multiple (only).

We will work both ways. You decide which way is easier.

Let's try some:

Example 3:

Evaluate $\int\left(x^2+1\right)^{12}\left(2x\right)dx$ using u-substitution as well as by pattern recognition.

Example 4:

Evaluate $\int 3x^3 \sin\left(2x^4+1\right)dx$ using *u*-substitution as well as by pattern recognition.

When integrating by pattern recognition, you will collect no more than three different types of scalar/constant multiples out in front of your antiderivative:

- constant multiples that were there in the original integrand (I call these "riders"),
- constant multiples generated from an integration rule like the power rule (I call these "rule" constants), and finally,
- constant multiples that "correct" any unwanted constant multiple generated by the derivative of the "inside function." These values will always be the reciprocal of the unwanted value (I call these "corrections.")

After integrating, you can combine all of these scalar multiples to get your final answer. Oh, and don't forget your $+C$

Example 5:

Evaluate $\int \frac{(x+1)\sqrt[3]{x^2+2x}}{\pi}dx$

Example 6:

Each of the following have the same inside function, but a different outside function, and hence, a different rule of integration. Evaluate each.

(a) $\int 5\sin(2x)e^{\cos(2x)}dx$

(b) $\int 7\sin(2x)\sqrt[3]{\cos^2(2x)}dx$

(c) $\int 2\sin(2x)\cos^2(2x)\,dx$

(d) $\int(\sin 2x)\sin(\cos 2x)\,dx$

(e) $\int \frac{2\sin 2x}{3\sqrt[5]{\cos^2 x-\sin^2 x}}dx$

(f) $\int 11\sin(2x)\cos(2x)\,dx$

Example 7:

Evaluate $\int \sec^2 x\tan x\,dx$ two different ways. Show the antiderivatives are equivalent, but for a constant.

Example 8:
Evaluate

(a) $\int 4x^2 5^{x^3+7} \sec^2\left(5^{x^3+7}\right) dx$

(b) $\int 22x^2 \sin\left(5x^3\right) e^{\cos\left(5x^3\right)} dx$

We already know the following trig antiderivatives:

Example 9:
Evaluate the following:

(a) $\int \sin x dx$ (b) $\int \cos x dx$ (c) $\int \sec^2 x dx$ (d) $\int \csc^2 x dx$ (e) $\int \sec x \cdot \tan x dx$ (f) $\int \csc x \cdot \cot x dx$

We can expand our antidifferentiation repertoire by memorizing the integrals of the other four trig functions . . . but what ARE they?

Example 10:
Find each of the following by being clever, then memorize the results.

(a) $\int \tan x dx$ (b) $\int \cot x dx$ (c) $\int \sec x dx$ (d) $\int \csc x dx$

You can now handle integrals like the following . . .

Example 11:
Evaluate the following:

(a) $\int 5x^2 \tan\left(x^3+1\right)dx$

(b) $\int 2e^{-x} \sec\left(e^{-x}\right)dx$

Let's try some random integrals now to integrate all our integration methods.

Example 12:
Evaluate the following:

(a) $\int \frac{x}{x^2-4}dx$

(b) $\int \frac{5}{x\ln x}dx$

(c) $\int \frac{\sqrt[3]{\ln^2 x}}{4x}dx$

(e) $\int \frac{\csc(\ln x)}{ex}dx$

(d) $\int \frac{\arctan x}{1+x^2}dx$

There are some options when you are evaluating a definite integral . . .

Example 13:

Evaluate the following two ways, using pattern recognition and *u*-substitution: $\int_0^1 x\left(x^2+1\right)^3 dx$

Example 14:
Review the following antiderivatives:

(a) $\int \frac{dx}{\sqrt{1-x^2}} =$ (b) $\int \frac{dx}{1+x^2} =$ (c) $\int \frac{dx}{x\sqrt{x^2-1}} =$

If these antiderivatives themselves have inside functions, it is much easier to memorize the following forms than to arduously algebraically manipulate it to look like the forms above.

Inverse Trig Integral forms: (MEMORIZE)

For some constant a and some function of x, u . . .

$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a}+C$ $\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a}+C$ $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}arc\sec\frac{|u|}{a}+C$

Example 15:
Evaluate each of the following. Compare and Contrast.

(a) $\int \frac{dx}{\sqrt{4-x^2}}$ (b) $\int \frac{x}{\sqrt{4-x^2}}dx$ (c) $\int \frac{dx}{2+9x^2}$ (d) $\int \frac{x}{2+9x^2}dx$

The arcsecant integration pattern can be a bit high-maintenance.

Example 16:
Evaluate the following:

(a) $\int \frac{9}{5x\sqrt{x^2-16}}dx$ (b) $\int \frac{dx}{x\sqrt{4x^2-9}}$ (c) $\int \frac{dx}{\sqrt{e^{2x}-1}}$

Sometimes, we need to algebraically alter the integrand to resemble something we recognize how to do. And sometimes, like parents of fighting siblings, we need to send each kid to his own room . . .

Example 17:

Evaluate the following: $\int \frac{x+2}{\sqrt{4-x^2}}\,dx$

Example 18:

Two of the Three integrals you can do. Evaluate the two you can do, and understand why you cannot do the third one.

(a) $\int \frac{3x^3}{\sqrt{5-x^4}}dx$ (b) $\int \frac{3x^2}{\sqrt{5-x^4}}dx$ (c) $\int \frac{3x}{\sqrt{5-x^4}}dx$

It's time to develop some more strategies of what to do when the integrand is unrecognizable . . .

Example 19:

Find each of the following, if possible.

(a) $\int \frac{3}{x^2+6x+9}\,dx$ (b) $\int \frac{3}{x^2+6x+10}\,dx$ (c) $\int \frac{3}{x^2+6x+8}\,dx$

Recall that we can use the integral to help us find the area of the region. In such cases, it is important to recognize the region, or to recognize where the region is positive and negative, because WE are in charge of making negative regions positive!!!

Example 20:

Find the area of the region bounded by the graph of $f(x)=\frac{1}{\sqrt{3x-x^2}}$, the x-axis, and the lines $x=\frac{3}{2}$ and $x=\frac{9}{4}$. Verify on your calculator.

Time for yet another algebraic strategy . . .

Example 21:
Find

(a) $\int \frac{x^2+x+1}{x^2+1}dx$

(b) $\int \frac{x^3-3x^2+4x-9}{x^2+3}dx$

Here are some Triggy ones . . .

Example 22:
Find
(a) $\int \sin x \cos x dx$ (b) $\int \sin^2 x \cos x dx$ (c) $\int \sin^2 x dx$ (d) $\int \sin^3 x dx$

Here's a simple one, serving as a confidence booster and a seque . . .

Example 23:
Find $\int \sqrt{2x-1} dx$

Sometimes we MUST use *u*-substitution.

Example 24:
Find the following.

(a) $\int x\sqrt{2x-1} dx$ (b) $\int \frac{x^2}{\sqrt{2x-1}} dx$

Example 24 (continued):

(c) $\int \frac{2x}{(x+1)^2}\,dx$ (d) $\int \frac{dx}{\sqrt{x}\sqrt{1-x}}$ (e) $\int \frac{dx}{1+\sqrt{2x}}$

Example 25:
Use *u*-substitution to answer the following:

(a) If $\int \frac{dx}{1+\cos x} = \tan\left(\frac{x}{2}\right) + C$, find $\int \frac{1}{3+3\cos\left(\frac{x}{4}\right)}\,dx$ exactly where C is a constant.

(b) If $\int_0^{\infty} \frac{x}{e^x+1}\,dx = \frac{\pi^2}{12}$, find $\int_0^{\infty} \frac{x}{e^{5x}+1}\,dx$ exactly.

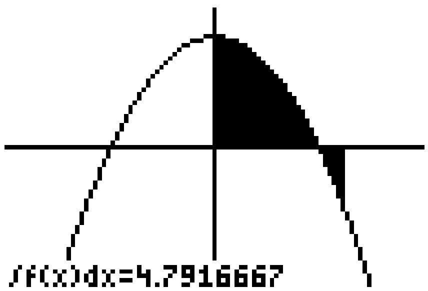
(c) If $\int_0^{\pi} \ln(a+b\cos(x))\,dx = 2\pi\ln(2)$, for some positive a and b, find the exact value of $\int_0^{\frac{\pi}{5}} \ln(a+b\cos(5x))\,dx$.

We will finish up with a discussion of how symmetry can help us evaluate an integral.

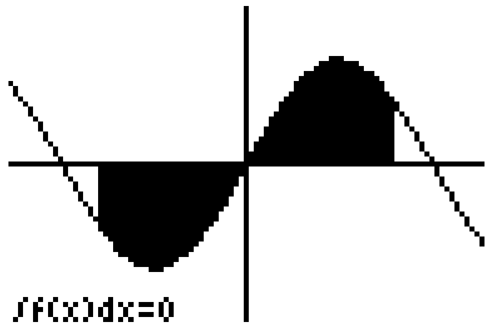
Integration of Even and Odd functions

Let f be an integrable function on a closed interval $[-a, a]$.

1. If f is an EVEN function, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$
2. If f is an ODD function, then $\int_{-a}^{a} f(x)\,dx = 0$



An Even function on a symmetrical interval



An Odd function on a symmetrical interval

Example 26:

Evaluate each of the following.

a) $\int_{-3}^{3}\left(\sin x + x^3 + \frac{x}{x^2+1}\right)dx$ b) $\int_{-\pi}^{\pi}\left(1 + x^2 + \cos x\right)dx$ c) $\int_{-1}^{1}\left(5x^5 + x^4 + 11x^3 + x^2 - 15x + 1\right)dx$