

§10.5—Surface Area

We have found areas, volumes, and even lengths using integration methods. Now we return to areas, but rather than working with flat surfaces, we will discover how integration methods can help us find surface areas of 3-D objects created by rotating a region about an axis of revolution.

Let the revolution begin!

Like any revolution, we must first arm ourselves with the requisite munitions for our task. In this case, we must review how to find the lateral surface area of the frustum of a right circular cone. If you already know this, continue on, we'll catch up with you soon.

Frustum is an interesting word, as well as an interesting shape. It's application in the real world can also lead to interesting fashion looks for canines. For example:

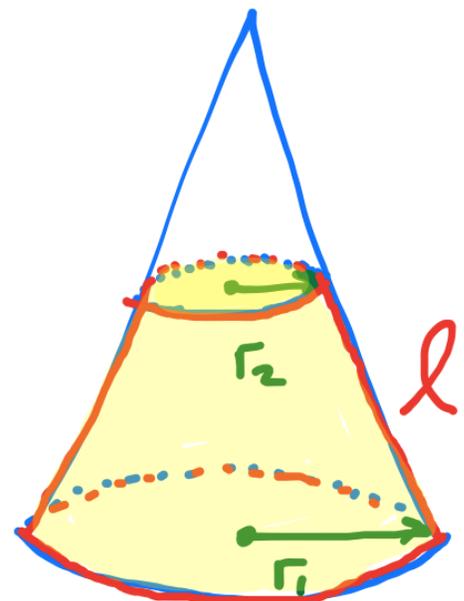


The **frustum** of a cone can be found by taking a right circular cone and truncating it parallel to the base at some height, creating two radii, the radius of the base, r_1 , and the radius of the truncated portion, r_2 . The slant height we will call ℓ .

The lateral surface area of the frustum of a cone is given by

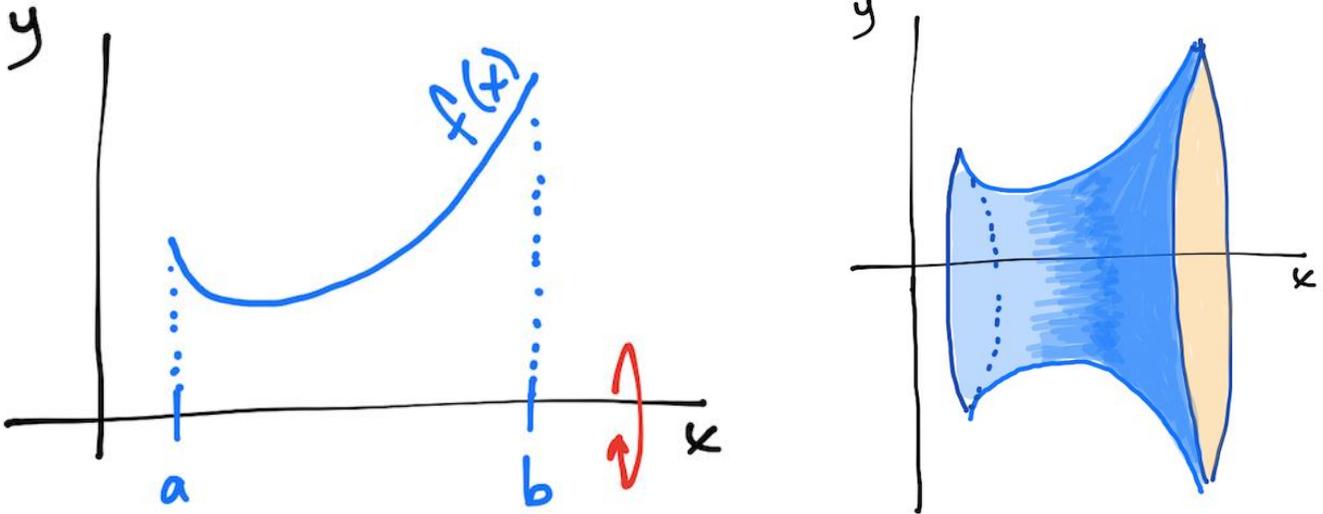
$$A = 2\pi r \ell$$

where r is the average radius, $r = \frac{r_1 + r_2}{2}$

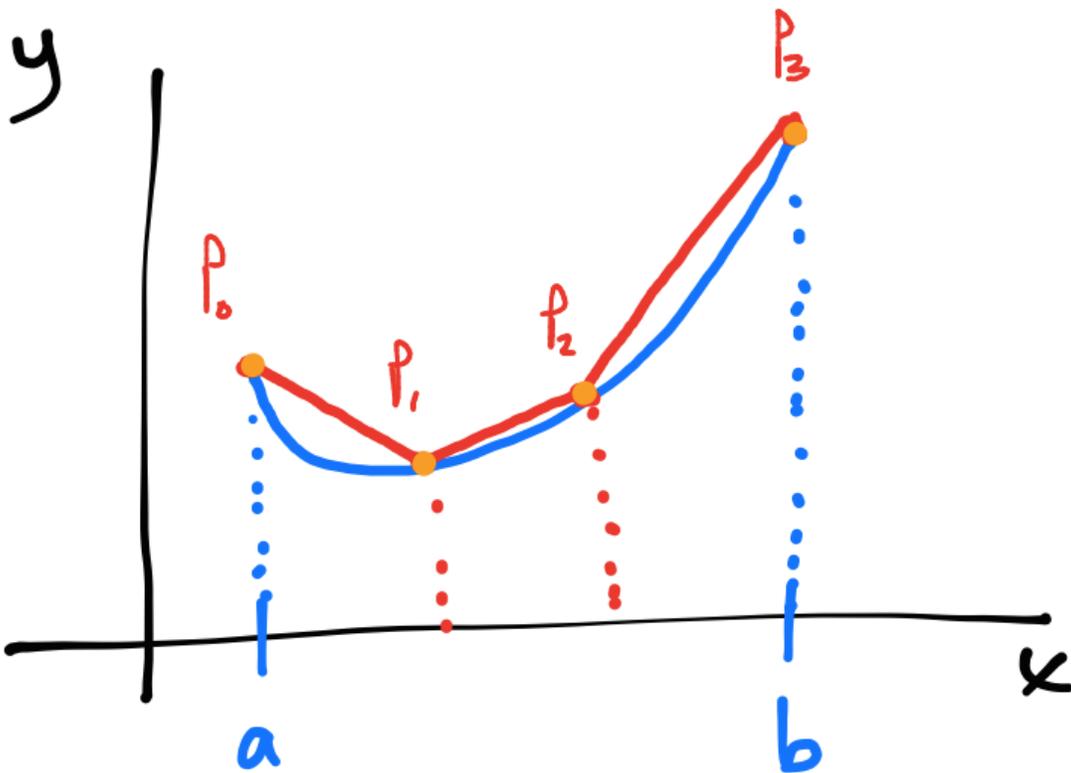


Here's where this formula comes in:

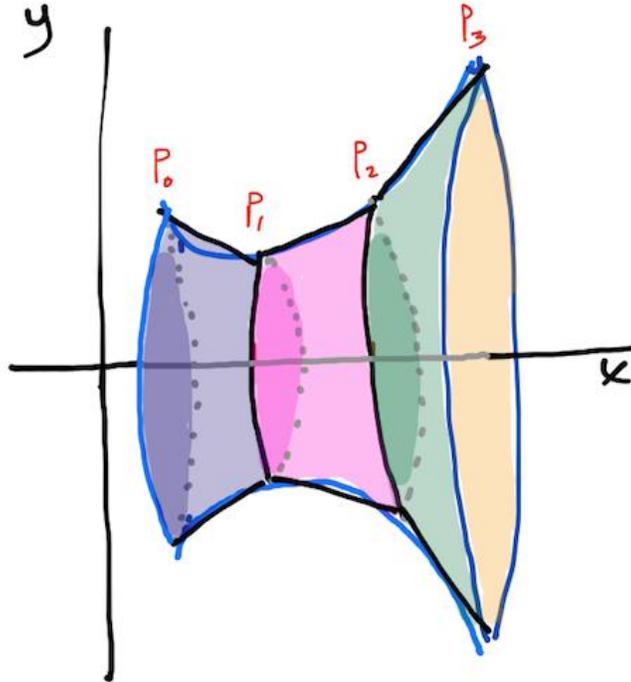
Suppose we take a continuous function $y = f(x)$ with continuous derivatives on the interval $[a, b]$ and rotate it about the x -axis generating a solid of revolution.



The integration formula for surface area can be derived in a similar fashion to the arc length integration formula from [8.4 notes](#). For demonstration purposes, we can divide the region from a to b into $n = 3$ partitions, drawing the line segments between consecutive points $P_0, P_1, P_2,$ and P_3 .



Rotating these segments around the x -axis creates three frustums of cones.



Returning to our formula for the surface area of a frustum, $A = 2\pi r\ell$, we can calculate, for each frustum, the average radius, r , and the slant height, ℓ , to obtain the formula.

NOTE: Similar to the Shell Method for volumes of revolution, since each “slice” is infinitely thin, the average radius converges to a single radius at some x -value.

Example 1:

Using the information given thus far, derive the integral formula for the surface area of a solid of revolution.

Surface Area of Revolution

If $f(x)$ is a function with continuous derivatives on an interval $[a, b]$, then the area, S , of the surface of revolution formed by revolving the graph of f about a horizontal axis is given by

$$S = 2\pi \int_a^b r(x) \sqrt{1 + [f'(x)]^2} dx$$

where $r(x)$ is the radius of rotation (the distance between the graph of $f(x)$ and the axis of revolution).

Example 2:

Verify the surface area of a sphere formula by finding the surface area of the solid formed by rotating

$f(x) = \sqrt{4 - x^2}$ about the x -axis.

Example 3:

Without a calculator, determine the surface area of revolution formed by revolving the graph of $f(x) = x^2$

on the interval $0 \leq x \leq \sqrt{2}$ about the y -axis.