Name Key Date Period

Worksheet 2.1—Tangent Line Problem

Show all work. No calculator permitted, except when stated.

Short Answer

1. Find the derivative function, $f'(x)$, for each of the following using the limit definition.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

(a) $f(x)=2x^2+3x-4$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\lim_{h\to 0}\frac{[2(x+h)^2+3(x+h)-4]-[2x^2+3x-4]}{h}$$

$$\lim_{h\to 0}\frac{\cancel{2x^2}+4xh+2h^2+\cancel{3x}+3h-\cancel{4}-\cancel{2x^2}-\cancel{3x}+\cancel{4}}{h}$$

$$\lim_{h\to 0}\frac{\cancel{h}(4x+2h+3)}{\cancel{h}}$$

$$4x+3$$

$$f'(x)=4x+3$$

(b) $f(x)=\dfrac{3}{x-1}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\lim_{h\to 0}\frac{\left(\frac{3}{x+h-1}\right)-\left(\frac{3}{x-1}\right)}{h}\left(\frac{(x+h-1)(x-1)}{(x+h-1)(x-1)}\right)$$

$$\lim_{h\to 0}\frac{3(x-1)-3(x+h-1)}{h(x+h-1)(x-1)}$$

$$\lim_{h\to 0}\frac{\cancel{3x}-\cancel{3}-\cancel{3x}-3\cancel{h}+\cancel{3}}{\cancel{h}(x+h-1)(x-1)}$$

$$\lim_{h\to 0}\frac{-3}{(x+h-1)(x-1)}$$

$$f'(x)=\frac{-3}{(x-1)^2}$$

(c) $f(x)=\sqrt{x-2}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\lim_{h\to 0}\frac{\sqrt{x+h-2}-\sqrt{x-2}}{h}\left(\frac{\sqrt{x+h-2}+\sqrt{x-2}}{\sqrt{x+h-2}+\sqrt{x-2}}\right)$$

$$\lim_{h\to 0}\frac{\cancel{x}+\cancel{h}-\cancel{2}-\cancel{x}+\cancel{2}}{\cancel{h}(\sqrt{x+h-2}+\sqrt{x-2})}$$

$$\lim_{h\to 0}\frac{1}{\sqrt{x+h-2}+\sqrt{x-2}}$$

$$f'(x)=\frac{1}{2\sqrt{x-2}}$$

2. Find the slope of the tangent lines to the graphs of the following functions at the indicated points. Use the ***alternate form.***

same as find derivative!

$$f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

(a) $f(x)=3-2x$ at $(-1,5)$

$c=-1$ $f(c)=5$

$$f'(-1)=\lim_{x\to -1}\frac{f(x)-f(-1)}{x-(-1)}$$

$$\lim_{x\to -1}\frac{3-2x-5}{x+1}$$

$$\lim_{x\to -1}\frac{-2(x+1)}{x+1}$$

$$f'(-1)=-2$$

(b) $g(x)=5-x^2$ at $x=2$

$c=2$ $g(c)=1$

$$g'(2)=\lim_{x\to 2}\frac{g(x)-g(2)}{x-2}$$

$$\lim_{x\to 2}\frac{5-x^2-1}{x-2}$$

$$\lim_{x\to 2}\frac{-(x^2-4)}{x-2}$$

$$\lim_{x\to 2}\frac{-(x+2)\cancel{(x-2)}}{\cancel{x-2}}$$

$$g'(2)=-4$$

Slope of tangent line $g'(2)$

3. Find the equation of the tangent line, in Taylor Form: $y = y_1 + m(x - x_1)$, for $g(x) = x^2 + 1$ at $(2,5)$. Use the ***modified form*** to find $g'(2)$.

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$$g'(2) = \lim_{h \to 0} \frac{g(2+h) - g(2)}{h}$$

$$\lim_{h \to 0} \frac{(2+h)^2 + 1 - ((2)^2 + 1)}{h}$$

$$\lim_{h \to 0} \frac{\cancel{4} + 4h + h^2 + \cancel{1} - \cancel{5}}{h}$$

$$\lim_{h \to 0} \frac{\cancel{h}(4+h)}{\cancel{h}}$$

$$g'(2) = 4$$

Taylor Form: $y = y_1 + m(x - x_1)$

$m = g'(2) = 4$

pt $(x_1, y_1) = (2,5)$

Taylor Form

$y = 5 + 4(x-2)$

Slope tangent line = derivative at x-value.

4. Find the equation of the tangent line, in Taylor Form: $y = y_1 + m(x - x_1)$, for $y = \sqrt{x} - 1$ at $c = 9$. Use the ***alternate form*** to find $y'(9)$.

$$y'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

$c = 9 \quad y(9) = \sqrt{9} - 1 = 2$

$$y'(9) = \lim_{h \to 0} \frac{y(9+h) - y(9)}{h}$$

$$\lim_{h \to 0} \frac{\sqrt{9+h} - 1 - 2}{h}$$

$$\lim_{h \to 0} \frac{\sqrt{9+h} - 3}{h} \left(\frac{\sqrt{9+h} + 3}{\sqrt{9+h} + 3} \right)$$

$$\lim_{h \to 0} \frac{\cancel{9} + \cancel{h} - \cancel{9}}{\cancel{h}(\sqrt{9+h} + 3)}$$

$$\lim_{h \to 0} \frac{1}{\sqrt{9+h} + 3}$$

$$y'(9) = \frac{1}{6}$$

Slope of tangent line at $x = 9$.

Taylor Form:

$m = y'(9) = \frac{1}{6}$

pt $(x_1, y_1) = (9, 2)$

Equation of Tangent Line in Taylor Form:

$y = 2 + \frac{1}{6}(x - 9)$

Need $f'(x)$

5. Find an equation of the line that is tangent to $f(x) = x^3$ and parallel to the line $3x - y + 1 = 0$. Remember, parallel lines have the same slope, but different base camps.

$3x - y + 1 = 0$

$y = 3x + 1$

slope = 3

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$\lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

$$\lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2)}{\cancel{h}}$$

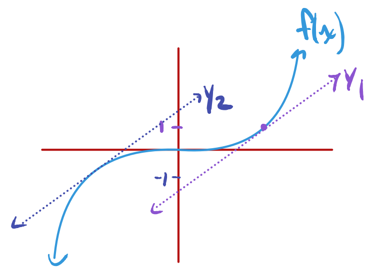
$f'(x) = 3x^2$

Slope tangent line needs to equal 3

$3x^2 = 3$

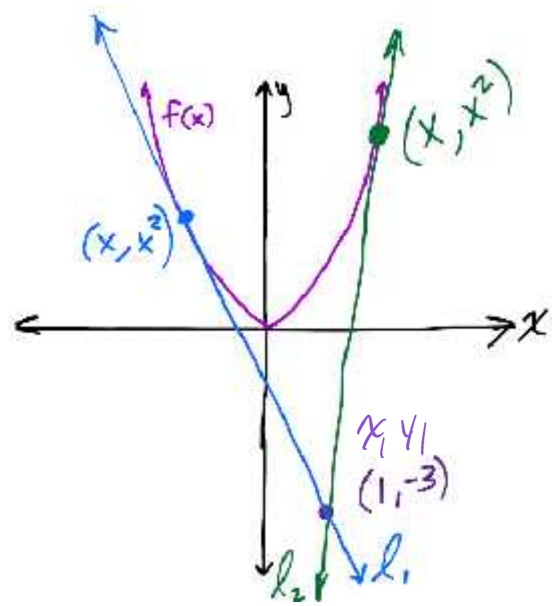
$x = \pm 1$, two eq'ns!

Visual



Eq'n 1	Eq'n 2
pt: $(1, f(1))$	pt $(-1, f(-1))$
$m = 3$	$m = 3$
$f(1) = 1$	$f(-1) = -1$
$y_1 = 1 + 3(x - 1)$	$y_2 = -1 + 3(x + 1)$

6. Find the equations of the two lines, ℓ_1 and ℓ_2, that are tangent to the graph of $f(x) = x^2$ if each pass through the point $(1, -3)$, as shown at right. Hint: equate two different expressions for finding the slope of a line. Solve the resulting equation.



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$\lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$\lim_{h \to 0} \frac{\cancel{h}(2x + h)}{\cancel{h}}$$

$f'(x) = 2x$

slope of line thru pt $(-1, 3)$

$$m_{\ell_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$(x-1)\, 2x = \frac{x^2 - -3}{x - 1}(x - 1)$$

$2x^2 - 2x = x^2 + 3$

$x^2 - 2x - 3 = 0$

$(x + 1)(x - 3) = 0$

$x + 1 = 0 \quad x - 3 = 0$

$x = -1 \quad x = 3$

Eq'n 1 (ℓ_1)

point $(-1, f(-1)) = (-1, 1)$

$m = f'(-1) = 2(-1) = -2$

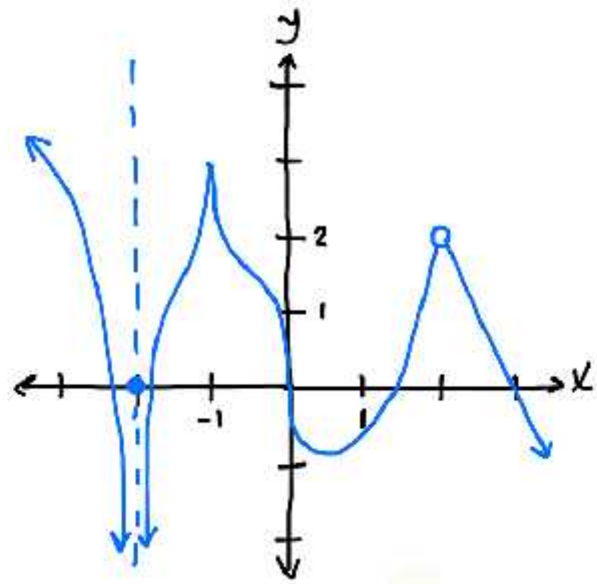
Taylor form: $y = 1 - 2(x + 1)$

Eq'n 2 (ℓ_2)

point: $(3, f(3)) = (3, 9)$

$m = f'(3) = 2(3) = 6$

Taylor form: $y = 9 + 6(x - 3)$



7. The graph of a function $f(x)$ is show above. For which value(s) of *x* is the graph of $f(x)$ not differentiable. In each case, explain why not.

Not diff'able at:

$x=-2$ b/c f is not cont. at $x=-2$ since $\lim_{x\to -2^-} f(x)=$ DNE

$x=-1$ b/c the graph of $f(x)$ has a cusp at $x=-1$ (ie: slopes tangent lines different)

$x=0$ b/c there is a vertical tangent at $x=0$ (ie: slope tangent line is infinite)

$x=2$ b/c f is not cont. at $x=2$ since $f(2)=$ DNE.

8. For each of the following, the limit represents $f'(c)$ for a function $f(x)$ and a number $x=c$. Find both *f* and *c*.

(a) $\lim_{h\to 0}\frac{[5-3(\cancel{1+}h)]-2}{h}$

$f(x)=5-3x$
$c=1$

(b) $\lim_{h\to 0}\frac{(\cancel{-2+}h)^3+8}{h}$

$f(x)=x^3$
$c=-2$

(c) $\lim_{x\to \underset{c}{6}}\frac{-x^2+36}{x-6}$

$f(x)=-x^2$
$c=6$

(d) $\lim_{x\to 9}\frac{2\sqrt{x}-6}{x-9}$

$f(x)=2\sqrt{x}$
$c=9$

→ Cannot use continuity as justification for not diff'able.

9. Using the alternate form, determine whether each of the following function is differentiable at the indicated point. Show the work that leads to your answer.

(a) $f(x) = \begin{cases} 5 - 4x, & x \le 0 \\ -2x^2, & x > 0 \end{cases}$ at $x = 0$

$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$

$\lim_{x \to 0^-} \frac{5 - 4x - (5 - 4(0))}{x}$

$\lim_{x \to 0^-} \frac{-4x}{x}$

-4

$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$

$\lim_{x \to 0^+} \frac{-2x^2 - (5 - 4(0))}{x}$

$\lim_{x \to 0^+} \frac{-2x^2 - 5}{x}$

DNE $(-\infty)$

$\therefore$ Since $-4 \neq$ DNE, f is not differentiable at $x = 0$.

(b) $f(x) = \begin{cases} (x - 1)^3, & x \le 1 \\ (x - 1)^2, & x > 1 \end{cases}$ at $x = 1$

$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}$

$\lim_{x \to 1^-} \frac{(x-1)^3 - (1-1)^3}{x - 1}$

$\lim_{x \to 1^-} (x-1)^2$

0

$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$

$\lim_{x \to 1^+} \frac{(x-1)^2 - (1-1)^3}{x - 1}$

$\lim_{x \to 1^+} x - 1$

0

$\therefore$ Since $0 = 0$, f is differentiable at $x = 1$.

10. **True or False. If false, explain why or give a counterexample.**

(a) The slope of the tangent line to the differentiable function *f* at the point $(2, f(2))$ is $\frac{f(2+h) - f(2)}{h}$.

False, slope of tangent line at $(2, f(2))$ is: $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$

(b) If a function is continuous at a point, then that function is differentiable at that point.

False, can be diff'able.

If there's a cusp or vertical tangent at that point it is not diff'able.

Note: If a fxn is diff'able then it's continuous is TRUE!

(c) If a function's slopes from both the right and the left at a point are the same, then that function is differentiable at that point.

False, fxn must be continuous.

If there's a jump or hole it's not diff'able (tangent slopes could be the same).

(d) If a function is differentiable at a point, then that function is continuous at that point.

100% True!

$D \Rightarrow C$ but $C \not\Rightarrow D$.

11. Using your **calculator** to zoooooom in, determine if $h(x)=\sqrt{x^2+0.0001}+0.99$ is locally linear at $x=0$. Give a reason for your answer. (Zoom WAAAAAY in!)

yes h(x) is locally linear (horiz. line) at $x=0$
and since h(x) is continuous at $x=0$, h(x) is also
diff'able at $x=0$.

Multiple Choice

E 12. A function will fail to be differentiable at all of the following except

(A) A vertical asymptote (B) A removable discontinuity (C) A cusp
(D) A vertical tangent line (E) A horizontal tangent line

slope $=0$

wk! $\frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}=x+2$

$$f(x)=\begin{cases}\frac{x^2-4}{x-2}, & x\neq 2 \quad 4\\ 1, & x=2 \quad 1\end{cases}$$

A 13. Let f be the function defined above. Which of the following statements about f are true?

I. $\lim_{x\to 2} f(x)$ exists ✓
II. f is continuous at $x=2$ ✗
III. f is differentiable at $x=2$ ✗

(A) I only (B) II only (C) III only (D) I and II only (E) I, II, and III

B 14. Let f be a differentiable function such that $f(2)=1$ and $f'(2)=4$. Let $T(x)$ be the equation of the tangent line to $f(x)$ at $x=2$. What is the value of $T(1.9)$?

(A) 0.4 (B) 0.6 (C) 0.7 (D) 1.3 (E) 1.4

point: $(x_1,y_1)=(2,1)$
$m=f'(2)=4$
Eq: $y=1+4(x-2)$
$y(1.9)=1+4(1.9-2)$
$=0.6$

C 15. Let f be a function such that $\overbrace{\lim_{h\to 0}\frac{f(7+h)-f(7)}{h}}^{f'(7)=5}=5$. Which of the followign must be true?

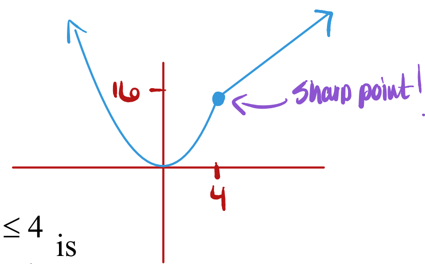
I. f is continuous at $x=7$ ✓ D ⇒ C

II. f is differentiable at $x=7$ ✓

III. The derivative of f is differentiable at $x=7$ (might be)*

(A) I only (B) II only (C) I and II only (D) I and III only (E) II and III only

* Would have to find $\lim_{h\to 0}\frac{f'(7+h)-f'(7)}{h}$ to determine whether its true.



B 16. At $x=4$, the function given by $h(x)=\begin{cases} x^2, & x\le 4 \\ 4x, & x>4 \end{cases}$ is

(A) Undefined

(B) Continuous but not differentiable

(C) Differentiable but not continuous

(D) Neither continuous nor differentiable

(E) Both continuous and differentiable

$\lim_{x\to 4^-} f(x)=16=f(4)=\lim_{x\to 4^+} f(x)$

continuous at $x=4$.

Diff'able Check

$\lim_{x\to 4^-}\frac{f(x)-f(4)}{x-4}$ $\lim_{x\to 4^+}\frac{f(x)-f(4)}{x-4}$

$\lim_{x\to 4^-}\frac{x^2-16}{x-4}$ $\lim_{x\to 4^+}\frac{4x-16}{x-4}$

$\lim_{x\to 4^-}\frac{(x+4)(x-4)}{x-4}$ $\lim_{x\to 4^+}\frac{4(x-4)}{x-4}$

$8 \neq 4$

Not diff'able!