Name Key Date______________________ Period______

Worksheet 2.3—Differentiation Rules

Show all work. No Calculator unless stated otherwise.

Short Answer

For 1-3, using correct notation (always), find the derivatives of the following functions. Simplify early and often!! Be sure to consider rewriting each term in the correct form first.

1. $f(x) = -2x^3 - x^2 + 4x - 7$

$f'(x) = -6x^2 - 2x + 4$

2. $g(x) = \dfrac{1}{x} - 3\sin x + \dfrac{4x + 2\sqrt{x}}{3\sqrt[3]{x}}$

$g(x) = x^{-1} - 3\sin x + \dfrac{4x}{3x^{1/3}} + \dfrac{2x^{1/2}}{3x^{1/3}}$

$g(x) = x^{-1} - 3\sin x + \dfrac{4}{3}x^{2/3} + \dfrac{2}{3}x^{1/6}$

$g'(x) = -1x^{-2} - 3\cos x + \dfrac{8}{9}x^{-1/3} + \dfrac{1}{9}x^{-5/6}$

$g'(x) = -\dfrac{1}{x^2} - 3\cos x + \dfrac{8}{9\sqrt[3]{x}} + \dfrac{1}{9\sqrt[6]{x^5}}$

3. $y = 4x^2(3-2x)^2$

$y = 4x^2(9 - 12x + 4x^2)$

$y = 36x^2 - 48x^3 + 16x^4$

$y' = \dfrac{dy}{dx} = 72x - 144x^2 + 64x^3$

For 4-6, using correct notation (never not always), evaluate each of the following with respect to the indicated variable. Simplify early and often!! Be sure to consider rewriting each term in the correct form first.

4. $\dfrac{d}{dt}\left[\sqrt{3t} - 6\sqrt[4]{t} - 4\cos t - \pi\right] =$

$\dfrac{d}{dt}\left[\sqrt{3}t^{1/2} - 6t^{1/4} - 4\cos t - \pi\right]$

$\dfrac{\sqrt{3}}{2}t^{-1/2} - \dfrac{3}{2}t^{-3/4} + 4\sin t$

$\dfrac{\sqrt{3}}{2\sqrt{t}} - \dfrac{3}{2\sqrt[4]{t^3}} + 4\sin t$

5. $\dfrac{d}{dx}\left[\left(\dfrac{x}{x^2+1}\right)^{-1}\right] =$

$\dfrac{d}{dx}\left[\dfrac{x^2+1}{x}\right]$

$\dfrac{d}{dx}\left[x + x^{-1}\right]$

$1 - x^{-2}$

$1 - \dfrac{1}{x^2}$ or

$\dfrac{x^2-1}{x^2}$

6. $\dfrac{d}{dm}\left[\dfrac{m^{-1} + m^{-2}}{m^{-3}}\right] =$

$\dfrac{d}{dm}\left[\dfrac{\frac{1}{m} + \frac{1}{m^2}}{\frac{1}{m^3}}\left(\dfrac{m^3}{m^3}\right)\right]$

$\dfrac{d}{dm}\left[\dfrac{m^2 + m}{1}\right]$

$\dfrac{d}{dm}\left[m^2 + m\right]$

$2m + 1$

7. For $f(x)=(x^2+2x)(x+1)$, find $f'(x)$. Remember to simplify early and often!

$f(x)=x^3+2x^2+x^2+2x$

$f(x)=x^3+3x^2+2x$

$f'(x)=3x^2+6x+2$

Need slope $= f'(x)$

(a) find the equation of the tangent line, in Taylor form, to the graph of f at $x=1$.

Need point $(x, f(x))$

from above,
$f'(x)=3x^2+6x+2$

$f'(1)=11$

$f(1)=6$

Equation of Tangent line
$y=y_1+m_T(x-x_1)$
$y=6+11(x-1)$

(b) find the equation of the normal line, in Taylor form, to the graph of f at $x=1$.

perpend. to tangent line.

from a: $f'(1)=11,\ f(1)=6$

$m=\frac{-1}{f'(1)}=-\frac{1}{11}$

eq. normal line:
$y=y_1+m_N(x-x_1)$
$y=6-\frac{1}{11}(x-1)$

(c) Using your equation from part (b), find the x-intercept of the normal line. Show the work that leads to your answer.

from (b), Normal line: $y=6-\frac{1}{11}(x-1)$

$11\cdot 0=\left(6-\frac{1}{11}(x-1)\right)11$

$0=66-x+1$

$x=67$

x-int: $(67,0)$.

For 8-10, determine the point(s) (if any) at which the graph of the following functions have horizontal tangent lines. Justify. horiz. tangent lines occur when the derivative $=0$.

8. $y=x+\sin x,\ 0\le x<2\pi$

$y'=\frac{dy}{dx}=1+\cos x$

$0=1+\cos x$

$-1=\cos x$

$x=\pi$

$y(\pi)=\pi$

y has a horiz. tangent line at the point (π,π).

9. $y=\sqrt{3}x+2\cos x,\ x\in[0,2\pi)$

$y'=\frac{dy}{dx}=\sqrt{3}-2\sin x$

$\sqrt{3}-2\sin x=0$

$\sin x=\frac{\sqrt{3}}{2}$

$x=\frac{\pi}{3},\ \frac{2\pi}{3}$

$y\left(\frac{\pi}{3}\right)=\sqrt{3}\left(\frac{\pi}{3}\right)+2\cos\frac{\pi}{3}$

$=\frac{\sqrt{3}\pi}{3}+1$

$y\left(\frac{2\pi}{3}\right)=\sqrt{3}\left(\frac{2\pi}{3}\right)+2\cos\frac{2\pi}{3}$

$=\frac{2\sqrt{3}\pi}{3}-1$

y has horiz. tangent lines at the points $\left(\frac{\pi}{3},\frac{\sqrt{3}\pi}{3}+1\right)$ & $\left(\frac{2\pi}{3},\frac{2\sqrt{3}\pi}{3}-1\right)$

10. $f(x)=\frac{2}{x^2}$

$f(x)=2x^{-2}$

$f'(x)=-4x^{-3}$

$f'(x)=\frac{-4}{x^3}$

$\frac{-4}{x^3}\neq 0$

Since $f'(x)\neq 0$, $f(x)$ does not have any points w/ horiz. tangent lines.

At a point of tangency we know a fxn and its tangent share ① y-value & ② slope value.

For 11 & 12, find the value of k such that the given line is tangent to the graph of the given function.

11. $f(x) = x^2 - kx$, line $y = 4x - 9$

① y-value

$f(x) = y$

$x^2 - kx = 4x - 9$

② slopes

$f'(x) = y'$

$2x - k = 4$

$2x = 4 + k$

$x = \frac{1}{2}(4+k)$

$\left(\frac{1}{2}(4+k)\right)^2 - k\left(\frac{1}{2}(4+k)\right) = 4\left(\frac{1}{2}(4+k)\right) - 9$

$4\left[\frac{1}{4}(4+k)^2 - k\left(2 + \frac{1}{2}k\right)\right] = \left[2(4+k) - 9\right]4$

$16 + 8k + k^2 - 8k - 2k^2 = 8(4+k) - 36$

$-k^2 + 16 = 32 + 8k - 36$

$-k^2 + 16 = 8k - 4$

$+k^2 - 16 \quad +k^2 - 16$

$0 = k^2 + 8k - 20$

$0 = (k+10)(k-2)$

$k + 10 = 0 \quad k - 2 = 0$

$\boxed{k = -10 \quad k = 2}$

12. $f(x) = k\sqrt{x} = kx^{1/2}$, line $y = x + 4$

① y-value

$f(x) = y$

$kx^{1/2} = x + 4$

$(2x^{1/2})(x^{1/2}) = x + 4$

$2x = x + 4$

$x = 4$

$k = 2x^{1/2}$

$= 2(4)^{1/2}$

$\boxed{k = 4}$

② slope

$f'(x) = y'$

$\frac{1}{2}kx^{-\frac{1}{2}} = 1$

$2x^{1/2} \cdot \frac{k}{2x^{1/2}} = 1 \cdot 2x^{1/2}$

$k = 2x^{1/2}$

Questions 13-16 are True of False. If False, either explain why, rewrite it to make it true, or provide a counterexample.

13. If $f'(x) = g'(x)$, then $f(x) = g(x)$.

False!

Let $f(x) = 2x^2 + 3$ and $g(x) = 2x^2 + 5$

$f'(x) = 4x$ and $g'(x) = 4x$

but $f(x) \neq g(x)$.

14. If $f(x) = g(x) + C$, then $f'(x) = g'(x)$.

True

If $f(x) = g(x) + C$ then $\frac{d}{dx}[f(x)] = \frac{d}{dx}[g(x) + C]$

$f'(x) = \frac{d}{dx}g(x) + \frac{d}{dx}C$

$= g'(x) + 0$

$= g'(x)$

15. If $y = \pi^3$, then $\frac{dy}{dx} = 3\pi^2$.

False!

$y = \pi^3$ is a constant

$\frac{dy}{dx}(\pi^3) = 0$

16. If $f(x) = \frac{1}{x^n}$, then $f'(x) = \frac{1}{nx^{n-1}}$

False!

If $f(x) = \frac{1}{x^n} = x^{-n}$

then $f'(x) = -nx^{-n-1}$

$= -nx^{-(n+1)}$

$= \frac{-n}{x^{n+1}} \neq \frac{1}{nx^{n-1}}$

17. (Calculator permitted) A priceless Faberge egg is dropped from the top of a building that is 1362 feet tall. The egg's height in feet at time t seconds is given by $h(t) = -16t^2 + 1362$.

Avg RoC!

(a) Find the average velocity, in ft/sec, of the egg on the interval $[1,2]$ seconds. Show the work that leads to your answer (always).

$$\text{Avg RoC} = \frac{h(2) - h(1)}{2-1} = \frac{[-16(2)^2 + 1362] - [-16(1)^2 + 1362]}{1} = 1298 - 1346 = -48$$

The avg ROC of $h(t)$ on $[1,2]$ is -48 ft/sec.

(b) Based on your answer in part (a), fill in the blanks so that the following sentence verbally describes the result found above.

"On the interval from $t =$ __1__ seconds to $t =$ __2__ seconds, the egg's height is __decreasing__ (increasing/decreasing), on average, by __48__ feet per second."

$v(t) = h'(t)$

(c) Find the instantaneous velocity of the egg at $t = 2$ seconds. Show the work that leads to your answer (always).

$h(t) = -16t^2 + 1362$

$h'(t) = -32t$

$h'(t) = v(t)$

$v(2) = -32(2)$

$v(2) = -64$

The instantaneous velocity of the egg at time $t = 2$ sec is -64 ft/sec

(d) Based on your answer in part (c), fill in the blanks so that the following sentence verbally describes the result found above.

"At $t = 2$ __seconds__, the egg's height is __decreasing__ (increasing/decreasing), by 64 __ft/sec__."

(e) After how many seconds will the egg hit the ground? Show the work that leads to your answer.

$h(t) = -16t^2 + 1362$

$h(t) = 0 = -16t^2 + 1362$

$16t^2 = 1362$

$t = \pm\sqrt{\frac{1362}{16}}$ $\quad t = \sqrt{\frac{1362}{16}}$

The egg will hit the ground after $\sqrt{\frac{1362}{16}}$ sec. which is approx 9.226 sec.

(f) Find the velocity, in ft/sec, of the egg as it hits the ground.

from (c) $v(t) = h'(t) = -32t$

from (e) $t = 9.226\ldots$ sec $= A$

$v(A) = -295.242$ ft/sec

18. Find the values of a and/or b ($a \neq 0$), if they exist, such that f is differentiable for all x. As always, show the work that leads to your answer.

To be diff'able $\forall x$ need: $\lim\limits_{x \to c^-} f(x) = f(c) = \lim\limits_{x \to c^+} f(x)$ AND $\lim\limits_{x \to c^-} f'(x) = \lim\limits_{x \to c^+} f'(x)$

(a) $f(x) = \begin{cases} ax^3, & x \le 2 \\ x^2 + b, & x > 2 \end{cases}$

Continuity: $\lim\limits_{x \to 2^-} f(x)$ $\quad f(2)$ $\quad \lim\limits_{x \to 2^+} f(x)$

$\lim\limits_{x \to 2^-} ax^3$ $\quad a(2)^3$ $\quad \lim\limits_{x \to 2^+} x^2 + b$

$a(2)^3$ $\quad 8a$ $\quad 2^2 + b$

$4 + b$

Derivatives $\lim\limits_{x \to 2^-} f'(x)$ $\quad \lim\limits_{x \to 2^+} f'(x)$

$\lim\limits_{x \to 2^-} 3ax^2$ $\quad \lim\limits_{x \to 2^+} 2x$

$3a(-2)^2$ $\quad 2(2)$

$12a$ $\quad 4$

To be diff'able at $x = 2$

Need $8a = 4 + b$ and $4 = 12a$

$4 = 12a$

$\boxed{\frac{1}{3} = a}$

$8\left(\frac{1}{3}\right) = 4 + b$

$\boxed{-\frac{4}{3} = b}$

(b) $f(x) = \begin{cases} \cos x, & x < \frac{\pi}{2} \\ ax + b, & x \ge \frac{\pi}{2} \end{cases}$

Continuity $\lim\limits_{x \to \frac{\pi}{2}^-} f(x)$ $\quad f\left(\frac{\pi}{2}\right)$ $\quad \lim\limits_{x \to \frac{\pi}{2}^+} f(x)$

$\cos \frac{\pi}{2}$ $\quad \frac{\pi}{2}a + b$ $\quad \frac{\pi}{2}a + b$

0

Derivatives $\lim\limits_{x \to \frac{\pi}{2}^-} f'(x)$ $\quad \lim\limits_{x \to \frac{\pi}{2}^+} f'(x)$

$\lim\limits_{x \to \frac{\pi}{2}^-} -\sin x$ $\quad \lim\limits_{x \to \frac{\pi}{2}^+} a$

$-\sin \frac{\pi}{2}$ $\quad a$

-1

To be diff'able $\forall x$, $a = -1$ and $0 = \frac{\pi}{2}a + b$

$0 = \frac{\pi}{2}(-1) + b$

$\frac{\pi}{2} = b$

$\boxed{a = -1 \text{ \& } b = \frac{\pi}{2}}$

(c) $f(x) = \begin{cases} \sin x, & x \le 0 \\ ax, & x > 0 \end{cases}$

Continuity $\lim\limits_{x \to 0^-} f(x)$ $\quad f(0)$ $\quad \lim\limits_{x \to 0^+} f(x)$

$\lim\limits_{x \to 0^-} \sin x$ $\quad 0$ $\quad \lim\limits_{x \to 0^+} ax$

0 $\quad 0$

$\therefore$ Since $0 = 0 = 0$, $f(x)$ is cont. $\forall x$.

Derivative $\lim\limits_{x \to 0^-} f'(x)$ $\quad \lim\limits_{x \to 0^+} f'(x)$

$\lim\limits_{x \to 0^-} \cos x$ $\quad \lim\limits_{x \to 0^+} a$

1 $\quad a$

To be diff'able $\forall x$, $a = 1$.

(d) $f(x) = \begin{cases} ax^2, & x \le 1 \\ b\sqrt{x}, & x > 1 \end{cases}$

Continuity $\lim\limits_{x \to 1^-} f(x)$ $\quad f(1)$ $\quad \lim\limits_{x \to 1^+} f(x)$

$\lim\limits_{x \to 1^-} ax^2$ $\quad a$ $\quad \lim\limits_{x \to 1^+} bx^{1/2}$

a $\quad b$

Derivatives $\lim\limits_{x \to 1^-} f'(x)$ $\quad \lim\limits_{x \to 1^+} f'(x)$

$\lim\limits_{x \to 1^-} 2ax$ $\quad \lim\limits_{x \to 1^+} \frac{1}{2}bx^{-1/2}$

$2a$ $\quad \frac{1}{2}b$

To be diff'able $\forall x$ need $a = b$

and $2a = \frac{1}{2}b$.

$2a = \frac{1}{2}a$

$a = 0$ (given $a \neq 0$)

$\therefore$ Since $a \neq 0$, $\exists$ no such values for a & b such that $f'(x)$ is diff'able at $x = 1$.

Multiple Choice

D 19. The function $f(x) = 3\sqrt[3]{x^2} + x - 1$ is differentiable for which values of x?

(A) all real numbers (B) $x \in [0, \infty)$ (C) $x \in (0, \infty)$ (D) for all $x \neq 0$ (E) $(-\infty, 0)$

$f(x) = 3x^{2/3} + x - 1$ how do you know there's a cusp?

$f'(x) = 2x^{-1/3} + 1$

$= \frac{2}{\sqrt[3]{x}} + 1$ f is diff'able $\forall x, x \neq 0$.

$x \neq 0$!

vertical tangent!

E 20. The function $f(x) = |x+4| - \sqrt[5]{x^3} + \frac{1}{x-3}$ is differentiable for all x-values except

I. $x = -4$ ✓

II. $x = 0$ ✓

III. $x = 3$ ✓

sharp edge $x = -4$

VA at $x = 3$!

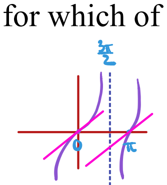
(A) I only (B) I and II only (C) I and III only (D) III only (E) I, II, and III

C 21. On the interval $[0, 2\pi)$, for which of the following x-values is the function $f(x) = \tan x$ not differentiable?

I. $x = 0$

II. $x = \pi$ continuous!

III. $x = \frac{3\pi}{2}$



(A) I only (B) II only (C) III only (D) I and II only (E) I and III only
(F) II and III only (G) I, II, and III (H) None of these