Name Key Date______________ Period______

Worksheet 2.4—Product & Quotient Rules

Show all work. No calculator permitted unless otherwise stated.

Short Answer

1. Find the derivative of each function using correct notation (never not always). Show all steps, including rewriting the original function as well as **simplifying your final answer s by combining like terms and/or factoring out common factors.** (except part (d)).

(a) $h(t) = 2t\cos t + t^2\sin t$

$h'(t) = 2\cos t + 2t(-\sin t) + 2t\sin t + t^2\cos t$

$= 2\cos t - \cancel{2t\sin t} + \cancel{2t\sin t} + t^2\cos t$

$= \cos t(2 + t^2)$

(b) $f(x) = 2x^2\cot x$

$f'(x) = 4x\cot x + 2x^2(-\csc^2 x)$

$= 4x\cot x - 2x^2\csc^2 x$

$= 2x(2\cot x - x\csc^2 x)$

(c) $f(x) = \dfrac{\tan x}{\sin x + 1}$

$f'(x) = \dfrac{(\sin x+1)(\sec^2 x) - \tan x(\cos x)}{(\sin x+1)^2}$

$= \dfrac{\sin x\sec^2 x + \sec^2 x - \sin x}{(\sin x+1)^2}$

$= \dfrac{\sec^2 x(\sin x+1) - \sin x}{(\sin x+1)^2}$

(d) $f(x) = \dfrac{x\sec x}{x^2+1}$

$f'(x) = \dfrac{(x^2+1)[\sec x + x\sec x\tan x] - x\sec x(2x)}{(x^2+1)^2}$

$= \dfrac{\sec x(x^2+1)(1+x\tan x) - 2x^2\sec x}{(x^2+1)^2}$

(e) $f(x) = \cot x\csc x$

$f'(x) = -\csc^2 x\cdot\csc x + \cot x(-\csc x\cot x)$

$= -\csc^3 x - \csc x\cot^2 x$

$= -\csc x(\csc^2 x + \cot^2 x)$

$= -\csc x(\csc^2 x + (\csc^2 x - 1))$

$= -\csc x(2\csc^2 x - 1)$

(f) $h(x) = \csc^2 x = (\csc x)(\csc x)$

$h'(x) = -\csc x\cdot\cot x(\csc x) + \csc x(-\csc x\cot x)$

$= -\csc^2 x\cot x - \csc^2 x\cot x$

$= -2\csc^2 x\cot x$

2. If $f(x)=\sin x(\sin x+\cos x)$, find the equation of the tangent line at $x=\frac{\pi}{4}$.

$f\left(\frac{\pi}{4}\right)=\sin\frac{\pi}{4}\left(\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\right)$

$=\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)$

$=\frac{\sqrt{2}}{2}\left(\sqrt{2}\right.$

$f\left(\frac{\pi}{4}\right)=1$

$f'(x)=\cos x(\sin x+\cos x)+\sin x(\cos x-\sin x)$

$=\sin x\cos x+\cos^2 x+\sin x\cos x-\sin^2 x$

$=\cos^2 x+2\sin x\cos x-\sin^2 x$

$=\sin 2x+\cos 2x$

$f'(x)=\sin 2x+\cos 2x$

$f'\left(\frac{\pi}{4}\right)=\sin 2\left(\frac{\pi}{4}\right)+\cos 2\left(\frac{\pi}{4}\right)$

$=1+0$

$=1$

Eq. tangent line

$y=1+1\left(x-\frac{\pi}{4}\right)$

3. Find the equation of the normal line to $f(x)=(x-1)(x^2+1)$ at the point where $f(x)$ crosses the x-axis.

$f(x)=0=(x-1)(x^2+1)$

$x-1=0$

$x=1$

$f'(x)=(1)(x^2+1)+(x-1)(2x)$

$f'(1)=(1)^2+1+((1)-1)(2(1))$

$=2$

$m_T=2 \quad m_N=-\frac{1}{2}$

Eq Normal line

$y=0-\frac{1}{2}(x-1)$

$y=-\frac{1}{2}(x-1)$

4. (Calculator Permitted) Determine the x-coordinates at which the graph of the function has a horizontal tangent line.

(a) $f(x)=\dfrac{x^2}{x-1}$

$f'(x)=\dfrac{(x-1)\cdot 2x-x^2(1)}{(x-1)^2}$

$=\dfrac{2x^2-2x-x^2}{(x-1)^2}$

$=\dfrac{x^2-2x}{(x-1)^2}$

$f'(x)=\dfrac{x(x-2)}{(x-1)^2}$

horiz. tangent occurs when $f'(x)=0$

$f'(x)=0$

$x(x-2)=0$

$x=0 \quad x-2=0$

$x=2$

$f(x)$ has a horiz. tangent at $x=0$ & $x=2$.

(b) $g(x)=x^2\sin x$, $-2\pi\le x\le 2\pi$

$g'(x)=2x\sin x+x^2\cos x$

$g'(x)=x(2\sin x+x\cos x)$

horiz. tangent when $g'(x)=0$

$g'(x)=0$

$x(2\sin x+x\cos x)=0$

$2\sin x+x\cos x=0$

$x=0 \quad x=\pm 5.086, \pm 2.288,$

$g(x)$ has a horiz. tangent at $x=0, \pm 2.288, \pm 5.086$

5. Find the equation(s) of the tangent line(s) to the graph of $y=\dfrac{x+1}{x-1}$ that are parallel to the line $2y+x=6$.

Parallel to $2y+x=6$

$y=-\frac{1}{2}x+3$

Slope of tangent, $m=-\frac{1}{2}$

$y'(x)=-\frac{1}{2}$

$$y'(x)=\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}$$

$$=\frac{x-1-x-1}{(x-1)^2}$$

$$=\frac{-2}{(x-1)^2}$$

$$y'(x)=2\cdot -\frac{1}{2}\cdot(x-1)^2=\frac{-2}{(x-1)^2}\cdot 2(x-1)^2$$

$$-1(x-1)^2=-4$$

$$x^2-2x+1=4$$

$$x^2-2x-3=0$$

$$x^2-2x-3=0$$

$$(x-3)(x+1)=0$$

$x-3=0$ $x+1=0$

$x=3$ $x=-1$

$y(3)=\frac{(3)+1}{(3)-1}=2$ $y(-1)=\frac{(-1)+1}{(-1)-1}=0$

Tangent 1: $y=2-\frac{1}{2}(x-3)$

Tangent 2: $y=0-\frac{1}{2}(x+1)$

6. The volume of a right circular cylinder is given by $V=\pi r^2 h$. If the radius of such a cylinder is given by $r=\sqrt{t+2}$ and its height is $h=\dfrac{\sqrt{t}}{2}$, where t is time in seconds and the dimensions are in inches.

(a) Find an equation for the volume, $V(t)$, of the right circular cylinder as a function of time.

$$V=\pi r^2 h$$

$$V(t)=\pi\left(\sqrt{t+2}\right)^2\left(\frac{\sqrt{t}}{2}\right)$$

$$=\pi(t+2)\cdot\frac{\sqrt{t}}{2}$$

$$V(t)=\frac{\pi}{2}\sqrt{t}(t+2)$$

(b) Find the rate of change of volume with respect to time, $V'(t)=\dfrac{dV}{dt}$.

from a, $V(t)=\frac{\pi}{2}\sqrt{t}(t+2)$

$$V'(t)=\frac{\pi}{2}\cdot\frac{1}{2}t^{-1/2}(t+2)+\frac{\pi}{2}\sqrt{t}(1)$$

$$=\frac{\pi(t+2)}{4\sqrt{t}}+\frac{\pi\sqrt{t}}{2}\left(\frac{2\sqrt{t}}{2\sqrt{t}}\right)$$

$$=\frac{\pi t+2\pi+2\pi t}{4\sqrt{t}}$$

$$=\frac{3\pi t+2\pi}{4\sqrt{t}}$$

$$V'(t)=\frac{\pi(3t+2)}{4\sqrt{t}}$$

(c) How fast is the volume of the cylinder changing when $t=1$?

from b, $V'(t)=\dfrac{\pi(3t+2)}{4\sqrt{t}}$

$$V'(1)=\frac{\pi(3(1)+2)}{4\sqrt{(1)}}$$

$$=\frac{5\pi}{4}$$

At time, $t=1$ sec, the volume is changing at a rate of $\frac{5\pi}{4}$ inches per second.

7. If the normal line to the graph of a function f at the point $(1,2)$ passes through the point $(-1,1)$, then what is the value of $f'(1)$? (Hint: Think Algebra I)

Normal line goes through $(1,2)$ & $(-1,1)$

$m_N = \frac{2-1}{1--1} = \frac{1}{2}$

$f'(1)$ = slope of tangent line at $x=1$

tangent line $\perp$ normal line

$f'(1) = -2$

8. Find the following by being cleverly clever.

(a) $\frac{d^{999}}{dx^{999}}[\cos x] = \frac{d^3}{dx^3}[\cos x] = \sin x$

$d^0 = \cos x$
$d^1 = -\sin x$
$d^2 = -\cos x$
$d^3 = \sin x$

$999 \div 4 = 249$, $3 = R$

(b) $\frac{d^4}{dx^4}\left[\frac{1}{x}\right] = \frac{d^4}{dx^4}\left[x^{-1}\right] =$

$\frac{d}{dx}[x^{-1}] = -x^{-2}$

$\frac{d^2}{dx^2}[x^{-1}] = 2x^{-3}$

$\frac{d^3}{dx^3}[x^{-1}] = -6x^{-4}$

$\frac{d^4}{dx^4}[x^{-1}] = 24x^{-5} = \frac{24}{x^5}$

Note the Recursive pattern!

$\frac{d^n}{dx^n}\left[x^{-1}\right] = (-1)^n \cdot n! \, x^{-(n+1)}$

Multiple Choice

A 9. If $y = \frac{2-x}{3x+1}$, then $\frac{dy}{dx} =$

(A) $-\frac{7}{(3x+1)^2}$ (B) $\frac{6x-5}{(3x+1)^2}$ (C) $-\frac{9}{(3x+1)^2}$ (D) $\frac{7}{(3x+1)^2}$ (E) $\frac{7-6x}{(3x+1)^2}$

$\frac{dy}{dx} = \frac{(3x+1)(-1)-(2-x)(3)}{(3x+1)} = \frac{-3x-1-6+3x}{(3x+1)^2} = \frac{-7}{(3x+1)^2}$

For questions 10-13, use the chart below, which gives selected values for differentiable functions $f(x)$ and $g(x)$ and their derivatives.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	5	-4
1	3	2	3	-3
2	5	3	1	-2
3	10	4	0	-1

B 10. If $h(x) = f(x) + 2g(x)$, then $h'(3) =$

(A) -2 (B) 2 (C) 7 (D) 8 (E) 10

$h'(x) = f'(x) + 2g'(x)$

$h'(3) = f'(3) + 2g'(3)$

$= 4 + 2(-1)$

$= 2$

B 11. If $h(x) = f(x) \cdot g(x)$, then $h'(2) =$

(A) -20 (B) -7 (C) -6 (D) -1 (E) 13

$h'(x) = f'(x)g(x) + f(x)g'(x)$

$h'(2) = f'(2)g(2) + f(2)g'(2)$

$= (3)(1) + (5)(-2)$

$= -7$

E 12. If $h(x) = \dfrac{1}{g(x)}$, then $h'(1) =$

(A) $-\dfrac{1}{2}$ (B) $-\dfrac{1}{3}$ (C) $-\dfrac{1}{9}$ (D) $\dfrac{1}{9}$ (E) $\dfrac{1}{3}$

$h'(x) = \dfrac{g(x) \cdot 0 - 1 g'(x)}{g^2(x)}$ $h'(1) = \dfrac{-g'(1)}{g^2(1)} = \dfrac{-(-3)}{(3)^2} = \dfrac{1}{3}$

$= \dfrac{-g'(x)}{g^2(x)}$

C 13. If $h(x) = \dfrac{f(x)}{g(x)}$, then $h'(0) =$

(A) $-\dfrac{13}{25}$ (B) $-\dfrac{1}{4}$ (C) $\dfrac{13}{25}$ (D) $\dfrac{13}{16}$ (E) $\dfrac{22}{25}$

$h'(x) = \dfrac{g(x) \cdot f'(x) - f(x)g'(x)}{g^2(x)}$

$h'(0) = \dfrac{g(0) \cdot f'(0) - f(0)g'(0)}{g^2(0)} = \dfrac{(5)(1) - (2)(-4)}{(5)^2} = \dfrac{13}{25}$