Name KEY Date__________ Period______

Worksheet 4.10—Derivatives of Log Functions & LOG DIFF

Show all work. No calculator unless otherwise stated.

1. Find the derivative of each function, given that a is a constant

(a) $y = x^a$

$\boxed{\frac{dy}{dx} = ax^{a-1}}$

(b) $y = a^x$

$\boxed{\frac{dy}{dx} = a^x \cdot \ln a}$

(c) $y = x^x$

$\ln y = x \ln x$

$\frac{d}{dx}[\ln y] = \frac{d}{dx}[x \ln x]$

$\frac{1}{y}\frac{dy}{dx} = (1)(\ln x) + (x)(\frac{1}{x})$

$\frac{dy}{dx} = [\ln x + 1]y$

$\boxed{\frac{dy}{dx} = (\ln x + 1)x^x}$

(d) $y = a^a$

$\boxed{0}$, a^a is a constant

2. Find the derivative of each. Remember to simplify early and often

(a) $\frac{d}{dx}\left[e^{2\ln x}\right] =$

$\frac{d}{dx}\left[e^{\ln x^2}\right]$

$\frac{d}{dx}\left[x^2\right]$

$\boxed{2x}$

(b) $\frac{d}{dx}\left[\log_a a^{\sin x}\right] =$

$\frac{d}{dx}[\sin x]$

$\boxed{\cos x}$

(c) $\frac{d}{dx}\left[\log_2 8^{x-5}\right] =$

$\frac{d}{dx}\left[\log_2 (2^3)^{x-5}\right]$

$\frac{d}{dx}\left[\log_2 2^{3x-15}\right]$

$\frac{d}{dx}[3x-15]$

$\boxed{3}$

3. For each of the following, find $\frac{dy}{dx}$. Remember to "simplify early and often."

(a) $y = \log_3 \frac{x\sqrt{x-1}}{2}$

$y = \log_3 x + \frac{1}{2}\log_3(x-1) - \log_3 2$

$\frac{dy}{dx} = \frac{1}{x\ln 3} + \frac{1}{2}\left(\frac{1}{(x-1)\ln 3}\right) - 0$

$\boxed{\frac{dy}{dx} = \frac{1}{x\ln 3} + \frac{1}{2\ln 3(x-1)}}$

(b) $y = x^{3/2}\log_2 \sqrt{x+1}$

$y = (x^{3/2})(\frac{1}{2}\log_2(x+1))$

$y = (\frac{1}{2}x^{3/2})(\log_2(x+1))$

$\frac{dy}{dx} = (\frac{3}{4}x^{1/2})(\log_2(x+1)) + (\frac{1}{2}x^{3/2})(\frac{1}{\ln 2(x+1)})$

$\boxed{\frac{dy}{dx} = \frac{3}{4}x^{1/2}\log_2(x+1) + \frac{x^{3/2}}{2\ln 2(x+1)}}$

(c) $y = \ln\left|\dfrac{\cos x}{\cos x - 1}\right|$

$y = \ln|\cos x| - \ln|\cos x - 1|$

$\dfrac{dy}{dx} = \left(\dfrac{1}{\cos x}\right)(-\sin x) - \left(\dfrac{1}{\cos x - 1}\right)(-\sin x)$

$\dfrac{dy}{dx} = -\dfrac{\sin x}{\cos x} + \dfrac{\sin x}{\cos x - 1}$

$\boxed{\dfrac{dy}{dx} = -\tan x + \dfrac{\sin x}{\cos x - 1}}$

(d) $y = \ln\left(\ln \dfrac{1}{x}\right)$

$\boxed{*\ln 1 = 0}$

$y = \ln(\ln 1 - \ln x)$

$y = \ln(-\ln x)$

$\dfrac{dy}{dx} = \dfrac{1}{-\ln x}\left(-\dfrac{1}{x}\right)$

$\boxed{\dfrac{dy}{dx} = \dfrac{1}{x\ln x}}$

(e) $y = \ln^3 x$

$y = (\ln x)^3$

$\dfrac{dy}{dx} = 3(\ln x)^2 \cdot \left(\dfrac{1}{x}\right)$

$\boxed{\dfrac{dy}{dx} = \dfrac{3\ln^2 x}{x}}$

(f) $y = x\ln x^2$

$y = x \cdot 2\ln x$

$y = (2x)(\ln x)$

$\dfrac{dy}{dx} = 2\ln x + 2x\left(\dfrac{1}{x}\right)$

$\boxed{\dfrac{dy}{dx} = 2\ln x + 2}$

(g) $y = \log_3(1 + x\ln x)$

$\dfrac{dy}{dx} = \dfrac{1}{\ln 3(1 + x\ln x)} \cdot \left(0 + (1)(\ln x) + (x)\left(\dfrac{1}{x}\right)\right)$

$\dfrac{dy}{dx} = \dfrac{1}{\ln 3(1 + x\ln x)}(\ln x + 1)$

$\boxed{\dfrac{dy}{dx} = \dfrac{1 + \ln x}{\ln 3(1 + x\ln x)}}$

(h) $y = \ln\sqrt[4]{\dfrac{4x-2}{3x+1}}$

$y = \ln\left(\dfrac{4x-2}{3x+1}\right)^{1/4}$

$y = \dfrac{1}{4}\ln\left(\dfrac{4x-2}{3x+1}\right)$

$y = \dfrac{1}{4}\left[\ln(4x-2) - \ln(3x+1)\right]$

$\dfrac{dy}{dx} = \dfrac{1}{4}\left[\left(\dfrac{1}{4x-2}\right)(4) - \left(\dfrac{1}{3x+1}\right)(3)\right]$

$\boxed{\dfrac{dy}{dx} = \dfrac{1}{4x-2} - \dfrac{3}{4(3x+1)}}$

4. Use implicit differentiation to find $\dfrac{dy}{dx}$.

(a) $x^2 - 3\ln y + y^2 = 10$

$\frac{d}{dx}\left[x^2 - 3\ln y + y^2\right] = \frac{d}{dx}[10]$

$2x - 3\left(\frac{1}{y}\right)\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx}\left(-\frac{3}{y} + 2y\right) = -2x$

$\frac{dy}{dx} = \frac{-2x}{\left(-\frac{3}{y} + 2y\right)} \cdot \frac{(-y)}{(-y)}$

$\boxed{\frac{dy}{dx} = \frac{2xy}{3 - 2y^2}}$

(b) $\ln xy + 5x = 30$

$\frac{d}{dx}\left[\ln x + \ln y + 5x\right] = \frac{d}{dx}[30]$

$\frac{1}{x} + \frac{1}{y}\left(\frac{dy}{dx}\right) + 5 = 0$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = -5 - \frac{1}{x}$

$\frac{dy}{dx} = \frac{\left(-5 - \frac{1}{x}\right)}{\frac{1}{y}} \cdot \frac{(-xy)}{(-xy)}$

$\boxed{\frac{dy}{dx} = \frac{5xy + y}{-x}}$

or $\frac{dy}{dx} = -\frac{5xy + y}{x}$

5. Find an equation of the tangent line to the graph of $x + y - 1 = \ln\left(x^2 + y\sqrt{2}\right)$ at $(1,0)$.

$\frac{d}{dx}[x + y - 1] = \frac{d}{dx}\left[\ln(x^2 + \sqrt{2}\,y)\right]$

$1 + \frac{dy}{dx} - 0 = \left(\frac{1}{x^2 + \sqrt{2}\,y}\right)\left(2x + \sqrt{2}\frac{dy}{dx}\right)$

At $(1,0)$:

$1 + \frac{dy}{dx} = \left(\frac{1}{1 + 0}\right)\left(2 + \sqrt{2}\frac{dy}{dx}\right)$

$1 + \frac{dy}{dx} = 2 + \sqrt{2}\frac{dy}{dx}$

$1 - 2 = \sqrt{2}\frac{dy}{dx} - \frac{dy}{dx}$

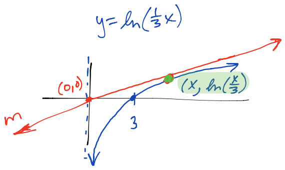
$\frac{dy}{dx}\left(\sqrt{2} - 1\right) = -1$

$\frac{dy}{dx} = \frac{-1}{\sqrt{2} - 1} = \boxed{\frac{1}{1 - \sqrt{2}} = m}$

eq: $\boxed{y = 0 + \frac{1}{1 - \sqrt{2}}(x - 1)}$

6. A line with slope m passes through the origin and is tangent to $y = \ln\left(\dfrac{x}{3}\right)$. What is the value of m?

$y = \ln\left(\frac{1}{3}x\right)$



pts: $(0,0)$ & $\left(x, \ln\left(\frac{x}{3}\right)\right)$

$m = \frac{\ln\left(\frac{x}{3}\right) - 0}{x - 0}$

$\boxed{m = \frac{1}{x}\ln\left(\frac{x}{3}\right)}$

$y = \ln x - \ln 3$

$\frac{dy}{dx} = \boxed{\frac{1}{x} = m}$

So $\frac{1}{x} = \frac{1}{x}\ln\left(\frac{x}{3}\right)$

$\frac{\ln\frac{x}{3}}{x} - \frac{1}{x} = 0$

$\frac{\ln\frac{x}{3} - 1}{x} = 0$

So $\ln\frac{x}{3} - 1 = 0$

$\ln\frac{x}{3} = 1$

$e \quad e$

$\frac{x}{3} = e$

$\boxed{x = 3e}$

So if $x = 3e$

$m = \frac{1}{x} = \boxed{\frac{1}{3e} = m}$

7. Find an equation for a line that is tangent to the graph of $y = e^x$ and goes through the origin.

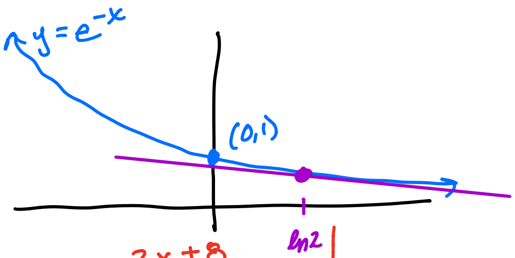
pt: $(0,0)$, $y' = e^x$

$y'(0) = e^0 = 1 = m$

eq: $y = 0 + 1(x - 0)$

or $\boxed{y = x}$

8. Find the point where the tangent line to the curve $y = e^{-x}$ is perpendicular to the line $-2x + y = 8$.



$y = 2x + 8$

$m = 2$, so

perpendicular slope is

$m_\perp = -\frac{1}{2}$.

So, the tangent line must have a slope of $-\frac{1}{2}$,

So $y' = -\frac{1}{2}$

$y = e^{-x}$

$y' = -e^{-x}$

So, $-e^{-x} = -\frac{1}{2}$

$e^{-x} = \frac{1}{2}$

$-x = \ln\left(\frac{1}{2}\right)$

$x = -\ln\left(\frac{1}{2}\right)$

$x = \ln\left(\frac{1}{2}\right)^{-1}$

$x = \ln 2$

So, the point is

$(\ln 2, y(\ln 2))$

$= (\ln 2, e^{-\ln 2}) = \left(\ln 2, \frac{1}{2}\right)$

Slope is $m = -\frac{1}{2}$

So, the equation is

$y = \frac{1}{2} - \frac{1}{2}(x - \ln 2)$

8. Use LOG DIFF:

(a) $\dfrac{d}{dx}\left[\sqrt[5]{\dfrac{(x-3)^4\left(x^2+1\right)}{(2x-5)^3}}\right]$

$y = \left(\dfrac{(x-3)^4(x^2+1)}{(2x-5)^3}\right)^{1/5}$

$y = \dfrac{(x-3)^{4/5}(x^2+1)^{1/5}}{(2x-5)^{3/5}}$

$\ln y = \frac{4}{5}\ln(x-3) + \frac{1}{5}\ln(x^2+1) - \frac{3}{5}\ln(2x-5)$

$\left(\frac{1}{y}\right)y' = \frac{4}{5}\left(\frac{1}{x-3}\right)(1) + \frac{1}{5}\left(\frac{1}{x^2+1}\right)(2x) - \frac{3}{5}\left(\frac{1}{2x-5}\right)(2)$

$y' = \left[\frac{4}{5(x-3)} + \frac{2x}{5(x^2+1)} - \frac{6}{5(2x-5)}\right]y$

$y' = \left[\frac{4}{5(x-3)} + \frac{2x}{5(x^2+1)} - \frac{6}{5(2x-5)}\right]\sqrt[5]{\frac{(x-3)^4(x^2+1)}{(2x-5)^3}}$

(b) If $y = x^{1/\ln x}$, find $\dfrac{dy}{dx}$.

$\ln y = \ln x^{1/\ln x}$

$\ln y = \frac{1}{\ln x} \cdot \ln x$

$\ln y = \frac{\cancel{\ln x}}{\cancel{\ln x}}$

$\ln y = 1$

$\frac{d}{dx}[\ln y] = \frac{d}{dx}[1]$

$\frac{1}{y}\frac{dy}{dx} = 0$

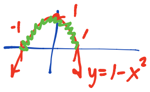
$\frac{dy}{dx} = 0(y)$

$\boxed{\frac{dy}{dx} = 0}$

9. Let $f(x) = \ln\left(1 - x^2\right)$.

(a) State the domain of *f*.

$1 - x^2 > 0$



the graph of $y = 1 - x^2$ is positive in between its x-intercepts of $x = -1, 1$

so $D_f : \{x \mid -1 < x < 1\}$ or $\boxed{D_f : (-1, 1)}$

(b) Find $f'(x)$.

$f'(x) = \left(\dfrac{1}{1-x^2}\right)(-2x)$

$\boxed{f'(x) = \dfrac{-2x}{1-x^2}}$

(c) State the domain of $f'(x)$.

** The domain of a function's derivative will either be the same as the original function, or a smaller subset of it, NOT LARGER!

* it would appear that the domain of $f'(x)$ is all x such that $1 - x^2 \neq 0$ (can't divide by zero) so $x \neq -1, 1$, BUT...

these values are already excluded from the domain of f, so $\boxed{D_{f'} : \{x \mid -1 < x < 1\}}$ (same domain as $f(x)$).

(d) Prove that $f''(x) < 0$ for all *x* in the domain of *f*.

$f'(x) = \dfrac{-2x}{1-x^2}$

$f''(x) = \dfrac{(1-x^2)(-2) - (-2x)(-2x)}{(1-x^2)^2}$

$f''(x) = \dfrac{-2 + 2x^2 - 4x^2}{(1-x^2)^2}$

$f''(x) = \dfrac{-2x^2 - 2}{(1-x^2)^2}$

$f''(x) = -2\left[\dfrac{x^2+1}{(1-x^2)^2}\right] < 0 \quad \forall x \in (-1, 1)$

Multiple Choice

B

10. Use the properties of logs to simplify, as much as possible, the expression:

$$\log_a 32 + \frac{4}{5}\log_a 4 - \frac{4}{5}\log_a 2 + \log_a \frac{1}{2^{\frac{14}{5}}}$$

(A) $\log_a 128$ (B) $\log_a 8$ (C) $\log_a 32$ (D) $\log_a 2^{-7}$ (E) 8

$\log_a 2^5 + \log_a (2^2)^{4/5} - \log_a 2^{4/5} + \log_a 2^{-14/5}$

$\log_a \left[\frac{2^5 \cdot 2^{8/5} \cdot 2^{-14/5}}{2^{4/5}}\right]$

$\log_a 2^{5+8/5-14/5-4/5}$

$\log_a 2^{5-\frac{10}{5}}$

$\log_a 2^{5-2}$

$\log_a 2^3$

$\boxed{\log_a 8}$

C

11. Simplify the expression as much as possible: $2^{5(\log_2 e)\ln x}$

(A) 5^x (B) e^{11} (C) x^5 (D) x^{10} (E) x^2

$2^{(5\ln x)(\log_2 e)}$

$2^{\log_2 e^{(5\ln x)}}$

$e^{5\ln x}$

$e^{\ln x^5}$

$\boxed{x^5}$

D

12. Which of the following is the domain of $f'(x)$ if $f(x) = \log_2(x+3)$?

(A) $x < -3$ (B) $x \le 3$ (C) $x \ne -3$ (D) $x > -3$ (E) $x \ge -3$

Domain of f: $x+3>0$
$x>-3$

$f' = \frac{1}{x+3}$ (apparent domain is $x \ne -3$, but domain of f' cannot be larger than that of f)

So $\boxed{D_{f'}: \{x \mid x>-3\}}$ (same as f)

A

13. If $f(x) = \left(x^2+1\right)^{(2-3x)}$, then $f'(1) =$

(A) $-\frac{1}{2}\ln(8e)$ (B) $-\ln(8e)$ (C) $-\frac{3}{2}\ln 2$ (D) $-\frac{1}{2}$ (E) $\frac{1}{8}$

LOG DIFF REQUIRED

$\ln f(x) = (2-3x)\ln(x^2+1)$

$\frac{1}{f(x)}\cdot f'(x) = (-3)\ln(x^2+1) + (2-3x)\left(\frac{1}{x^2+1}\right)(2x)$

$f'(x) = \left[-3\ln(x^2+1) + \frac{2x(2-3x)}{x^2+1}\right]f(x)$

$f'(1) = \left[-3\ln 2 + \frac{2(-1)}{2}\right]f(1)$

$= \left[-3\ln 2 - 1\right](2^{-1})$

$f'(1) = -\frac{1}{2}\left[3\ln 2 + \ln e\right]$

clever form of 1

$f'(1) = -\frac{1}{2}\left[\ln 2^3 + \ln e\right]$

$\boxed{f'(1) = -\frac{1}{2}\ln(8e)}$

B

14. Determine if $\lim_{x\to\infty}\left[\ln(2+5x) - \ln(2+3x)\right]$ exists, and if it does, find its value.

(A) $\ln\frac{1}{2}$ (B) $\ln\frac{5}{3}$ (C) $\ln\frac{3}{5}$ (D) $\ln 2$ (E) Does Not Exist

$\underbrace{\infty - \infty}_{\text{indeterminate form}}$ (Direct sub)

condense: $\lim_{x\to\infty} \ln\left(\frac{2+5x}{2+3x}\right)$

$\ln\left[\lim_{x\to\infty}\frac{5x+2}{3x+2}\right]$

$= \ln\left(\frac{5}{3}\right)$

E

15. Find the derivative of $f(t) = \frac{2\ln t}{3+\ln t}$.

(A) $f'(t) = \frac{2}{t(3+\ln t)^2}$ (B) $f'(t) = \frac{6\ln t}{(3+\ln t)^2}$ (C) $f'(t) = \frac{2\ln t}{3+\ln t}$

(D) $f'(t) = \frac{2}{t(3+\ln t)}$ (E) $f'(t) = \frac{6}{t(3+\ln t)^2}$ (F) $f'(t) = \frac{6\ln t}{3+\ln t}$

$f'(t) = \frac{(3+\ln t)\left(\frac{2}{t}\right) - (2\ln t)\left(\frac{1}{t}\right)}{(3+\ln t)^2}\left(\frac{t}{t}\right)$

$f'(t) = \frac{(3+\ln t)(2) - 2\ln t}{t(3+\ln t)^2}$

$f'(t) = \frac{6 + \cancel{2\ln t} - \cancel{2\ln t}}{t(3+\ln t)^2}$

$f'(t) = \frac{6}{t(3+\ln t)^2}$

B

16. Determine the derivative of *f* when $f(x) = x^{4x}$

(A) $f'(x) = (\ln x + 4)x^{4x}$ (B) $f'(x) = 4(\ln x + 1)x^{4x}$ (C) $f'(x) = 4(\ln x + 1)$

(D) $f'(x) = (\ln x + 1)x^{4x}$ (E) $f'(x) = x^{4(x-1)}$ (F) $f'(x) = 4x^{4(x-1)}$

LOG DIFF

$\ln f(x) = (4x)(\ln x)$

$\frac{1}{f(x)} \cdot f'(x) = 4\ln x + 4x\left(\frac{1}{x}\right)$

$f'(x) = [4\ln x + 4]x^{4x}$

$\boxed{f'(x) = 4(\ln x + 1)x^{4x}}$

C

17. Find the derivative of *f* when $f(x) = x[7\sin(\ln x) + 2\cos(\ln x)]$.

(A) $f'(x) = x[5\sin(\ln x) + 9\cos(\ln x)]$ (B) $f'(x) = 5\sin(\ln x) - 9\cos(\ln x)$

(C) $f'(x) = 5\sin(\ln x) + 9\cos(\ln x)$ (D) $f'(x) = 9\sin(\ln x) + 5\cos(\ln x)$

(E) $f'(x) = x[9\sin(\ln x) + 5\cos(\ln x)]$

$f'(x) = 1 \cdot [7\sin(\ln x) + 2\cos(\ln x)] + x\left[7\cos(\ln x)\left(\frac{1}{x}\right) - 2\sin(\ln x)\left(\frac{1}{x}\right)\right]$

$f'(x) = 7\sin(\ln x) + 2\cos(\ln x) + 7\cos(\ln x) - 2\sin(\ln x)$

$\boxed{f'(x) = 5\sin(\ln x) + 9\cos(\ln x)}$