Name KEY Date________________________ Period______

Worksheet 2.4—Product & Quotient Rules

Show all work. No calculator permitted unless otherwise stated.

Short Answer

1. Find the derivative of each function using correct notation (never not always). Show all steps, including rewriting the original function as well as **simplifying your final answer s by combining like terms and/or factoring out common factors.** (except part (d)).

(a) $h(t)=(2t)(\cos t)+(t^2)(\sin t)$

$h'(t) = 2\cos t + 2t(-\sin t) + 2t\sin t + t^2\cos t$

$h'(t) = 2\cos t - \underline{2t\sin t} + \underline{2t\sin t} + t^2\cos t$

$h'(t) = 2\cos t + t^2\cos t$

$h'(t) = \cos t(2+t^2)$

(b) $f(x)=(2x^2)(\cot x)$

$f'(x) = (4x)(\cot x) + (2x^2)(-\csc^2 x)$

$f'(x) = 4x\cot x - 2x^2\csc^2 x$

(c) $f(x)=\dfrac{\tan x}{\sin x+1}$

$f'(x) = \dfrac{(\sin x+1)(\sec^2 x)-(\tan x)(\cos x)}{(\sin x+1)^2}$

$= \dfrac{(\sin x+1)(\sec^2 x)-\left(\frac{\sin x}{\cancel{\cos x}}\right)(\cancel{\cos x})}{(\sin x+1)^2}$

$= \dfrac{(\sin x+1)(\sec^2 x)-\sin x}{(\sin x+1)^2}$

(d) $f(x)=\dfrac{(x)(\sec x)}{(x^2+1)}$

$f'(x)= \dfrac{(x^2+1)[(1)(\sec x)+(x)(\sec x\tan x)]-(x\sec x)(2x)}{(x^2+1)^2}$

$= \dfrac{(x^2+1)(\sec x)(1+x\tan x)-2x^2\sec x}{(x^2+1)^2}$

(e) $f(x)=(\cot x)(\csc x)$ $\quad * \; 1+\cot^2 x=\csc^2 x$

$f'(x)= (-\csc^2 x)(\csc x)+(\cot x)(-\csc x\cot x)$

$f'(x)= -\csc x(\csc^2 x+\cot^2 x)$

or $f'(x) = -\csc x(\csc^2 x + \underline{\csc^2 x - 1})$

$f'(x) = -\csc x(2\csc^2 x - 1)$

(f) $h(x)=\csc^2 x=(\csc x)(\csc x)$

$h'(x) = (-\csc x\cot x)(\csc x)+(\csc x)(-\csc x\cot x)$

$h'(x) = -\csc^2 x\cot x - \csc^2 x\cot x$

$h'(x) = -2\csc^2 x\cot x$

2. If $f(x)=\sin x(\sin x+\cos x)$, find the equation of the tangent line at $x=\frac{\pi}{4}$.

y-value: $f\left(\frac{\pi}{4}\right)=\left(\sin\frac{\pi}{4}\right)\left(\sin\frac{\pi}{4}+\cos\frac{\pi}{4}\right)$
$=\frac{\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right)$
$=\frac{\sqrt{2}}{2}(\sqrt{2})$
$=\frac{2}{2}$
$=1$

pt: $\left(\frac{\pi}{4},1\right)$

$f'(x)=\cos x(\sin x+\cos x)+\sin x(\cos x-\sin x)$
$=\cos x\sin x+\cos^2 x+\cos x\sin x-\sin^2 x$
$=2\sin x\cos x+\cos^2 x-\sin^2 x$
$=\sin 2x+\cos 2x$ (Double angle IDs)

Slope: $f'\left(\frac{\pi}{4}\right)=\sin\left(2\cdot\frac{\pi}{4}\right)+\cos\left(2\cdot\frac{\pi}{4}\right)$
$=\sin\frac{\pi}{2}+\cos\frac{\pi}{2}$
$=1+0$
$m=1$

$\boxed{\text{eq: } y=1+1\left(x-\frac{\pi}{4}\right)}$

3. Find the equation of the <u>normal</u> line to $f(x)=(x-1)(x^2+1)$ at the point where $f(x)$ crosses the x-axis.

find x-intercept
$f(x)=0$
$(x-1)(x^2+1)=0$
$x-1=0$ or $x^2+1=0$
$x=1$ No Real Solution
$f(1)=0$
pt $(1,0)$

$f'(x)=(1)(x^2+1)+(x-1)(2x)$
$f'(1)=(1^2+1)+(0)(2)$
$f'(1)=2$ = slope of tangent line
So Normal line slope $=-\frac{1}{2}$ (opposite reciprocal)

equation of normal line
$y=0-\frac{1}{2}(x-1)$
or
$y=-\frac{1}{2}x+\frac{1}{2}$

4. Determine the x-coordinates at which the graph of the function has a horizontal tangent line.

(a) $f(x)=\dfrac{x^2}{x-1}$

$f'(x)=\dfrac{(x-1)(2x)-(x^2)(1)}{(x-1)^2}$
$=\dfrac{x[2(x-1)-x]}{(x-1)^2}$
$=\dfrac{x[2x-2-x]}{(x-1)^2}$
$=\dfrac{x(x-2)}{(x-1)^2}$

$f'(x)=0$
when $x(x-2)=0$
$\boxed{x=0, x=2}$

(b) (Calculator Permitted) $g(x)=x^2\sin x,\ -2\pi\le x\le 2\pi$

$g'(x)=2x\sin x+x^2\cdot\cos x=0$
$x(2\sin x+x\cos x)=0$
$x=0$ or $2\sin x+x\cos x=0$
$\boxed{x=0}$ or (calculator)
y1 = $2\sin x+x\cos x$
y2 = 0
window: X[-2π, 2π]
Y[-1, 1]
find pts of intersection
2nd TRACE #5

$\boxed{\begin{array}{l}x=-5.086\\x=-2.288\\x=2.288\\x=5.086\end{array}}$

5. Find the equation(s) of the tangent line(s) to the graph of $y=\frac{x+1}{x-1}$ that are parallel to the line $2y+x=6$.

$2y=-x+6$
$y=-\frac{1}{2}x+3$
$m=-\frac{1}{2}$
(parallel lines have same slope, so $\frac{dy}{dx}=-\frac{1}{2}$ too!

$\frac{dy}{dx}=\frac{(x-1)(1)-(x+1)(1)}{(x-1)^2}=-\frac{1}{2}$

$\frac{x-1-x-1}{(x-1)^2}=-\frac{1}{2}$

$\frac{-2}{(x-1)^2}=-\frac{1}{2}$

$4=(x-1)^2$ (extract $\pm$ roots)

$x-1=2$ or $x-1=-2$

$x=3$ or $x=-1$

$y(3)=\frac{3+1}{3-1}=2$ pt: $(3,2)$ $m=-\frac{1}{2}$ eq: $y=2-\frac{1}{2}(x-3)$

$y(-1)=\frac{-1+1}{-1-1}=0$ pt: $(-1,0)$ $m=-\frac{1}{2}$ eq: $y=0-\frac{1}{2}(x+1)$

6. The volume of a right circular cylinder is given by $V=\pi r^2 h$. If the radius of such a cylinder is given by $r=\sqrt{t+2}$ and its height is $h=\frac{\sqrt{t}}{2}$, where t is time in seconds and the dimensions are in inches.

(a) Find an equation for the volume, $V(t)$, of the right circular cylinder as a function of time.

$V=\pi r^2 h$
$V=\pi\left(\sqrt{t+2}\right)^2\left(\frac{\sqrt{t}}{2}\right)$
$V(t)=\frac{\pi}{2}(t+2)(t^{1/2})$

(b) Find the rate of change of volume with respect to time, $V'(t)=\frac{dV}{dt}$.

$V'(t)=\frac{\pi}{2}\left[(1)(t^{1/2})+(t+2)\left(\frac{1}{2}t^{-1/2}\right)\right]$

$=\left(\frac{\pi}{2}\right)\left(\frac{1}{2}t^{-1/2}\right)\left[2t+(t+2)\right]$ *factor out least power, $t^{-1/2}$

$V'(t)=\frac{\pi}{4\sqrt{t}}(3t+2)$

$V'(t)=\frac{\pi(3t+2)}{4\sqrt{t}}$

(c) How fast is the volume of the cylinder changing when $t=1$?

$V'(1)=\left.\frac{dV}{dt}\right|_{t=1}=\frac{\pi(3(1)+2)}{4\sqrt{1}}$

$=\frac{\pi(5)}{4}$

$=\frac{5\pi}{4}$

7. If the normal line to the graph of a function f at the point $(1,2)$ passes through the point $(-1,1)$, then what is the value of $f'(1)$? (Hint: Think Algebra I)

$f'(1)$ is the slope of the tangent line at $(1,2)$

Normal slope is perpendicular to tangent slope.

$$\text{Normal slope} = \frac{y_2-y_1}{x_2-x_1} = \frac{1-2}{-1-1} = \frac{-1}{-2} = \frac{1}{2}$$

So $f'(1) = -2$ (opp. recip.)

8. Find the following by being cleverly clever.

(a) $\dfrac{d^{999}}{dx^{999}}[\cos x] =$

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{(4)}(x) = \cos x = f(x)$

$f^{(5)}(x) = -\sin x = f'(x)$

$\vdots$

(repeats every 4th deriv.)

So $4\overline{)999}$ = 249, $\frac{8}{19}$, $\frac{16}{39}$, $\frac{36}{3}$ remainder

So $\dfrac{d^{999}}{dx^{999}}[\cos x] = \dfrac{d^3}{dx^3}[\cos x] = f'''(x)$

$= \sin x$

(b) $\dfrac{d^4}{dx^4}\left[\dfrac{1}{x}\right] = \dfrac{d^4}{dx^4}\left[x^{-1}\right] =$

$\frac{d}{dx}[x^{-1}] = -x^{-2}$

$\frac{d^2}{dx^2}[x^{-1}] = \frac{d}{dx}[-x^{-2}] = 2x^{-3}$

$\frac{d^3}{dx^3}[x^{-1}] = \frac{d}{dx}[2x^{-3}] = -6x^{-4}$

$\frac{d^4}{dx^4}[x^{-1}] = \frac{d}{dx}[-6x^{-4}] = \boxed{24x^{-5}}$

Identify the recursive pattern

$\vdots$

$\underbrace{\frac{d^n}{dx^n}[x^{-1}] = (-1)^n \cdot n!\, x^{-(n+1)}}_{\text{this is the pattern for the } n^{th} \text{ derivative}}$

Multiple Choice

A 9. If $y = \dfrac{2-x}{3x+1}$, then $\dfrac{dy}{dx} =$

(A) $-\dfrac{7}{(3x+1)^2}$ (B) $\dfrac{6x-5}{(3x+1)^2}$ (C) $-\dfrac{9}{(3x+1)^2}$ (D) $\dfrac{7}{(3x+1)^2}$ (E) $\dfrac{7-6x}{(3x+1)^2}$

$$\frac{dy}{dx} = \frac{(3x+1)(-1)-(2-x)(3)}{(3x+1)^2} = \frac{-3x-1-6+3x}{(3x+1)^2} = \frac{-7}{(3x+1)^2}$$

For questions 10-13, use the chart below, which gives selected values for differentiable functions $f(x)$ and $g(x)$ and their derivatives.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	5	–4
1	3	2	3	–3
2	5	3	1	–2
3	10	4	0	–1

B 10. If $h(x) = f(x) + 2g(x)$, then $h'(3) =$

(A) -2 (B) 2 (C) 7 (D) 8 (E) 10

$h'(x) = f'(x) + 2g'(x)$
$h'(3) = f'(3) + 2g'(3)$
$= 4 + (2)(-1)$
$= 2$

B 11. If $h(x) = f(x) \cdot g(x)$, then $h'(2) =$

(A) -20 (B) -7 (C) -6 (D) -1 (E) 13

$h'(x) = f'(x)g(x) + f(x)g'(x)$
$h'(2) = f'(2)g(2) + f(2)g'(2)$
$= (3)(1) + (5)(-2)$
$= 3 - 10 = -7$

E 12. If $h(x) = \dfrac{1}{g(x)}$, then $h'(1) =$

(A) $-\dfrac{1}{2}$ (B) $-\dfrac{1}{3}$ (C) $-\dfrac{1}{9}$ (D) $\dfrac{1}{9}$ (E) $\dfrac{1}{3}$

$h'(x) = \dfrac{g(x)(0) - (1)(g'(x))}{(g(x))^2}$

$h(x) = \dfrac{-g'(x)}{(g(x))^2}$

$h'(1) = \dfrac{-g'(1)}{(g(1))^2}$
$= \dfrac{-(-3)}{(3)^2}$
$= \dfrac{3}{9}$
$= \dfrac{1}{3}$

C 13. If $h(x) = \dfrac{f(x)}{g(x)}$, then $h'(0) =$

(A) $-\dfrac{13}{25}$ (B) $-\dfrac{1}{4}$ (C) $\dfrac{13}{25}$ (D) $\dfrac{13}{16}$ (E) $\dfrac{22}{25}$

$h'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

$h'(0) = \dfrac{g(0)f'(0) - f(0)g'(0)}{(g(0))^2}$

$= \dfrac{(5)(1) - (2)(-4)}{5^2}$

$= \dfrac{5+8}{25}$

$= \dfrac{13}{25}$