Memorize!

$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}[\arccos x] = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$ $\frac{d}{dx}[\text{arccot}\, x] = \frac{-1}{1+x^2}$

$\frac{d}{dx}[\text{arcsec}\, x] = \frac{1}{|x|\sqrt{x^2-1}}$ $\frac{d}{dx}[\text{arccsc}\, x] = \frac{-1}{|x|\sqrt{x^2-1}}$

Name Key

Worksheet 2.8—Inverse & Inverse Trig Functions

Show all work. No calculator unless otherwise stated.

Short Answer

1. Find the derivative with respect to the appropriate variable. **Simplify your expression.**

(a) $y = \sec^{-1}(x^2)$

$\frac{d}{dx}[\sec(\ )] = \frac{1}{|\ |\sqrt{(\ )^2-1}}$

$\frac{d}{dx}: y' = \frac{1}{|x^2|\cdot\sqrt{(x^2)^2-1}}\cdot 2x$ $|x^2|>0$

$y' = \frac{2}{x\cdot\sqrt{x^4-1}}$

Note: if $x>0$, $\frac{dy}{dx}>0$ and if $x<0$, $\frac{dy}{dx}<0$.

(b) $y = s\sqrt{1-s^2} + \arccos s$

$y = s(1-s^2)^{1/2} + \arccos s$

$\frac{d}{dx}[\arccos x] = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{ds}: y' = 1\cdot(1-s^2)^{1/2} + \frac{1}{2}s(1-s^2)^{-1/2}(-2s) - \frac{1}{\sqrt{1-(s)^2}}$

$y' = \left(\frac{(1-s^2)^{1/2}}{(1-s^2)^{1/2}}\right)(1-s^2)^{1/2} - s^2(1-s^2)^{-1/2} - (1-s^2)^{-1/2}$

$y' = \frac{(1-s^2)^1 - s^2 - 1}{\sqrt{1-s^2}}$

$y' = \frac{-2s^2}{\sqrt{1-s^2}}$

(c) $y = \frac{1}{\arcsin(2x)} = (\arcsin 2x)^{-1}$

$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$

$y' = -1(\arcsin 2x)^{-2}\cdot\left(\frac{1}{\sqrt{1-(2x)^2}}\right)\cdot 2$

$y' = \frac{-2}{\arcsin^2 2x\sqrt{1-4x^2}}$

(d) $y = \cot^{-1}\sqrt{t-1}$

$y = \cot^{-1}(t-1)^{1/2}$

$\frac{d}{dx}[\cot^{-1} x] = \frac{-1}{1+x^2}$

$y' = \frac{-1}{1+((t-1)^{1/2})^2}\cdot\frac{1}{2}(t-1)^{-1/2}\cdot 1$

$y' = \frac{-1}{1+t-1}\cdot\frac{1}{2}(t-1)^{-1/2}\cdot 1$

$y' = \frac{-1}{2t\sqrt{t-1}}$

(e) $y = \csc^{-1}\frac{x}{2}$

$y = \csc^{-1}(\frac{1}{2}x)$

$\frac{d}{dx}[\csc^{-1} x] = \frac{-1}{|x|\sqrt{x^2-1}}$

$y' = \frac{-1}{|\frac{1}{2}x|\sqrt{(\frac{1}{2}x)^2-1}}\cdot\frac{1}{2}$

$y' = \frac{-1}{|x|\cdot\sqrt{\frac{1}{4}x^2-1}}\left(\frac{2}{2}\right)$

$y' = \frac{-2}{|x|\sqrt{x^2-4}}$

(f) $y = \sin(\arccos t)$

$\frac{d}{dx}[\arccos x] = \frac{-1}{\sqrt{1-x^2}}$

$y' = \cos(\arccos t)\cdot\left(\frac{-1}{\sqrt{1-t^2}}\right)$

$y' = \frac{-t}{\sqrt{1-t^2}}$

(g) $y = m\arctan m$

$\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$

$y' = 1\cdot\arctan m + m\left(\frac{1}{1+m^2}\right)$

$y' = \arctan m + \frac{m}{1+m^2}$

(h) $y = x\arcsin x + \sqrt{1+x^2}$

$y = x\arcsin x + (1+x^2)^{1/2}$

$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$

$y' = 1\cdot\arcsin x + x\cdot\left(\frac{1}{\sqrt{1-x^2}}\right) + \frac{1}{2}(1+x^2)^{-\frac{1}{2}}\cdot 2x$

$y' = \arcsin x + \frac{x}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1+x^2}}$

(i) $y = \frac{\arcsin 3x}{x}$

$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$

$y' = \frac{x\left(\frac{1}{\sqrt{1-(3x)^2}}\right)\cdot(3) - \arcsin(3x)\cdot 1}{x^2}$

$y' = \frac{3}{x\sqrt{1-9x^2}} - \frac{\arcsin(3x)}{x^2}$

(j) $y = \frac{1}{2}\left(x\sqrt{4-x^2} + 4\arcsin\left(\frac{x}{2}\right)\right)$

$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$

$y = \frac{1}{2}x(4-x^2)^{1/2} + 2\arcsin(\frac{1}{2}x)$

$y' = \frac{1}{2}(4-x^2)^{1/2} + \frac{1}{2}x\cdot\left(\frac{1}{2}(4-x^2)^{-1/2}(-2x)\right) + 2\left(\frac{1}{\sqrt{1-(\frac{1}{2}x)^2}}\cdot\frac{1}{2}\right)$

$y' = \frac{1}{2}(4-x^2)^{1/2} - \frac{1}{2}x^2(4-x^2)^{-1/2} + 2(4-x^2)^{-1/2}$

$y' = \frac{(4-x^2)^{1/2}}{(4-x^2)^{1/2}}\frac{(4-x^2)^{1/2}}{2} - \frac{x^2}{2(4-x^2)^{1/2}} + \frac{2}{(4-x^2)^{1/2}}\left(\frac{2}{2}\right)$

$y' = \frac{4-x^2-x^2+4}{2(4-x^2)^{1/2}}$

$y' = \frac{8-2x^2}{2(4-x^2)^{1/2}} = \frac{4-x^2}{(4-x^2)^{1/2}} = \sqrt{4-x^2}$

(k) $y = \sin^{-1} t + \cos^{-1} t$

$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$ $\frac{d}{dx}[\cos^{-1} x] = \frac{-1}{\sqrt{1-x^2}}$

$y' = \frac{1}{\sqrt{1-t^2}} + \frac{-1}{\sqrt{1-t^2}}$

$y' = 0$

1^{st} deriv@ $t=1$.

2. If a particle's position is given by $x(t)=\tan^{-1}(t^2)$, find the $\boxed{\text{particle's velocity at } t=1.}$

$\frac{d}{dx}[\tan^{-1}x]=\frac{1}{1+x^2}$

$x'(t)=\frac{1}{1+(t^2)^2}(2t)$

$x'(t)=\frac{2t}{1+t^4}$

$x'(1)=\frac{2}{2}=1$

Since $x'(t)=v(t)$,
$x'(1)=v(1)=1$.

3. (Unit Circle Time!) Find the equation for the tangent line to the graph of y at the indicated point.

(a) $y=\sec^{-1}x$ at $x=2$, $0\le y\le\frac{\pi}{2}$

point: $(2,y(2))=(2,\frac{\pi}{3})$

slope: $m=\frac{1}{2\sqrt{3}}$

$y'=\frac{1}{|x|\sqrt{(x)^2-1}}$

$y'(2)=\frac{1}{2\sqrt{4-1}}=\frac{1}{2\sqrt{3}}$

Eq of Tangent Line: $y=\frac{\pi}{3}+\frac{1}{2\sqrt{3}}(x-2)$

(b) $y=\sin^{-1}\left(\frac{x}{2}\right)$ at $x=\sqrt{3}$, $0\le y\le\frac{\pi}{2}$

point: $(\sqrt{3},y(\sqrt{3}))=(\sqrt{3},\frac{\pi}{3})$

slope: $m=1$

$y'=\frac{1}{\sqrt{1-(\frac{1}{2}x)^2}}\left(\frac{1}{2}\right)$

$y'=\frac{1}{\sqrt{4-x^2}}$

$y'(\sqrt{3})=\frac{1}{\sqrt{4-(\sqrt{3})^2}}=1$

Eq. of Tangent Line:
$y=\frac{\pi}{3}+1(x-\sqrt{3})$

4. A particle moves along the x-axis so that its position at any time $t\ge 0$ is given by $x(t)=\arctan t$.

(a) Prove that the particle is always moving to the right by analyzing the $\boxed{\text{velocity function.}}$

1^{st} derivative of $x(t)$.

$x(t)=\arctan t$

$v(t)=x'(t)=\frac{1}{1+t^2}$

Since $x'(t)>0$ $\forall t\ge 0$,
The particle is moving to the right $\forall t\ge 0$.

(b) Prove that the particle's velocity is always decreasing by analyzing the $\boxed{\text{acceleration function.}}$

2^{nd} derivative of $x(t)$.

$x(t)=\arctan t$

from a, $v(t)=x'(t)=\frac{1}{1+t^2}=(1+t^2)^{-1}$

$a(t)=v'(t)=x''(t)=-1(1+t^2)^{-2}\cdot(2t)$

$a(t)=\frac{-2t}{(1+t^2)^2}$

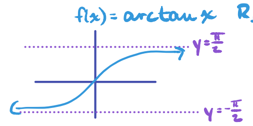
Since $a(t)<0$, $\forall t\ge 0$
the particle's velocity is decreasing, $\forall t\ge 0$.

(c) What is the $\boxed{\text{limiting position of the particle}}$ as t approaches infinity.

right end behavior of $x(t)$.

$\lim_{t\to\infty} x(t)=\frac{\pi}{2}$

Recall: $f(x)=\arctan x$ $R_f:(-\frac{\pi}{2},\frac{\pi}{2})$



5. If $f(x)=x^5+2x^3+x-1$ and $f(g(x))=x=g(f(x))$ find

(a) $f(1)$

$f(1) = 1+2+1-1 = 3$

$f: (1,3)$

$g: (3,1)$

(b) $g'(3)$.

$g'(3) = \frac{1}{f'(1)}$

$f'(x) = 5x^4+6x^2+1$

$f'(1) = 5+6+1 = 12$

$g'(3) = \frac{1}{12}$.

6. If $h(x)=\cos x+3x$, find $(h^{-1})'(h(0))$.

$h(0) = \cos(0)+3(0) = 1$

$h: (0,1)$

$h^{-1}: (1,0)$

$(h^{-1})'(h(0)) = (h^{-1})'(1) = \frac{1}{h'(0)}$

$h'(x) = -\sin x + 3$

$h'(0) = -\sin 0 + 3 = 3$

$(h^{-1})'(h(0)) = \frac{1}{3}$

7. Find the equation of the tangent line to the graph of $x^2+x\arctan y=y-1$ at $\left(-\frac{\pi}{4},1\right)$.

point: $\left(-\frac{\pi}{4},1\right)$

slope: $m = \frac{-2\pi}{8+\pi}$

$x^2 + x\arctan y = y-1$

$\frac{d}{dx}$: $2x + 1\cdot\arctan y + x\left(\frac{1}{1+y^2}\right)\frac{dy}{dx} = \frac{dy}{dx} - 0$

$2x + \arctan y = \frac{dy}{dx} - \frac{x}{1+y^2}\frac{dy}{dx}$

$2x + \arctan y = \left[1 - \frac{x}{1+y^2}\right]\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{2x+\arctan y}{1-\frac{x}{1+y^2}}$

$\left.\frac{dy}{dx}\right|_{\left(-\frac{\pi}{4},1\right)} = \frac{2\left(-\frac{\pi}{4}\right)+\arctan 1}{1-\frac{\left(-\frac{\pi}{4}\right)}{1+(1)^2}} = \frac{-\frac{\pi}{2}+\frac{\pi}{4}}{1+\frac{\pi}{8}}\left(\frac{8}{8}\right) = \frac{-2\pi}{8+\pi}$

Eq. of Tangent Line:

$y = 1 - \frac{2\pi}{8+\pi}\left(x+\frac{\pi}{4}\right)$

8. If $f(x)=\frac{1}{8}x-3$, find

(a) $f^{-1}(-3)$

$x = \frac{1}{8}y - 3$

$x+3 = \frac{1}{8}y$

$8x+24 = y$

$f^{-1}(x) = 8x+24$

$f(x) = -3 = \frac{1}{8}x - 3$

$x = 0$

$f^{-1}(-3) = 0$

(b) $(f^{-1})'(3.14159)$

$f'(x) = \frac{1}{8}$

$(f^{-1})'(3.14159) = \frac{1}{\left(\frac{1}{8}\right)} = 8$

(c) $(f^{-1})''(3.14159)$

$f''(x) = 0$

$(f^{-1})''(3.14159) = \text{DNE}$

Multiple Choice

C 9. Find the value of $f(1)$ when $f(x) = 5\sin^{-1}x + 6\tan^{-1}x$.

(A) 3π (B) 2π (C) 4π (D) $\frac{7\pi}{2}$ (E) $\frac{5\pi}{2}$

$f(1) = 5\sin^{-1}1 + 6\tan^{-1}1$

$f(1) = \frac{2}{2}5\left(\frac{\pi}{2}\right) + 6\left(\frac{\pi}{4}\right)$

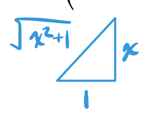
$f(1) = \frac{16\pi}{4} = 4\pi$

$= x$?

A 10. If $k(j(t)) = j(k(t))$ and $j(-3) = 2$, $k(-3) = 4$, $k(2) = -3$, $k'(-3) = \frac{2}{5}$, $k'(2) = \frac{4}{3}$, then $j'(-3) =$ $\frac{1}{k'(2)} = \frac{1}{4/3} = \frac{3}{4}$ $j: (-3, 2)$ $k: (2, -3)$

(A) $\frac{3}{4}$ (B) $-\frac{1}{3}$ (C) $\frac{5}{2}$ (D) $-\frac{5}{2}$ (E) $-\frac{3}{4}$

D 11. The expression $f(x) = \sin\left(\tan^{-1}x\right)$ is equivalent to which algebraic identity?



(A) $f(x) = \frac{1}{\sqrt{1+x^2}}$

(B) $f(x) = \sqrt{1+x^2}$ $f(x) = \frac{x}{\sqrt{x^2+1}}$

(C) $f(x) = \frac{x}{\sqrt{1-x^2}}$

(D) $f(x) = \frac{x}{\sqrt{1+x^2}}$

(E) $f(x) = \frac{1}{\sqrt{1-x^2}}$

E 12. Determine $f'(x)$ when $f(x) = \sin^{-1}\left(\frac{x}{\sqrt{6}}\right)$.

(A) $f'(x) = \frac{\sqrt{6}}{\sqrt{6+x^2}}$ $f'(x) = \frac{1}{\sqrt{1-\left(\frac{1}{\sqrt{6}}x\right)^2}}\left(\frac{1}{\sqrt{6}}\right)$

$f'(x) = \frac{1}{\sqrt{6\left(1-\frac{1}{6}x^2\right)}}$

(B) $f'(x) = \frac{x}{x^2+6}$ $f'(x) = \frac{1}{\sqrt{6-x^2}}$

(C) $f'(x) = \frac{x}{\sqrt{x^2-6}}$

(D) $f'(x) = \frac{\sqrt{6}}{\sqrt{6-x^2}}$

(E) $f'(x) = \frac{1}{\sqrt{6-x^2}}$

A ___ 13. Find the derivative of f when $f(x)=5\arcsin\frac{x}{5}+\sqrt{25-x^2}$ (this one requires a bit of algebraic finagling after you get the derivative, including factoring the radicand.)

(A) $f'(x)=\sqrt{\frac{5-x}{5+x}}$

(B) $f'(x)=\frac{x}{\sqrt{25-x^2}}$

(C) $f'(x)=\frac{5}{\sqrt{25-x^2}}$

(D) $f'(x)=\frac{1}{\sqrt{5+x}}$

(E) $f'(x)=\sqrt{\frac{5+x}{5-x}}$

$f'(x)=5\left(\frac{1}{\sqrt{1-\left(\frac{1}{5}x\right)^2}}\right)\left(\frac{1}{5}\right)+\frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x)$

$f'(x)=\frac{5}{\sqrt{25-x^2}}-\frac{x}{\sqrt{25-x^2}}$

$f'(x)=\frac{5-x}{\sqrt{25-x^2}}$

$f'(x)=\frac{(5-x)}{(5-x)^{1/2}(5+x)^{1/2}}$

$f'(x)=\sqrt{\frac{5-x}{5+x}}$

A ___ 14. Find the derivative of f when $f(x)=3\left(\sin^{-1}x\right)^2$.

(A) $f'(x)=\frac{6\sin^{-1}x}{\sqrt{1-x^2}}$

(B) $f'(x)=\frac{3\sin^{-1}x}{\sqrt{1-x^2}}$

(C) $f'(x)=\frac{6\sin^{-1}x}{1+x^2}$

(D) $f'(x)=\frac{6\cos^{-1}x}{\sqrt{1-x^2}}$

(E) $f'(x)=\frac{3\cos^{-1}x}{\sqrt{1-x^2}}$

$f'(x)=6(\sin^{-1}x)^1\cdot\left(\frac{1}{\sqrt{1-x^2}}\right)$

$f'(x)=\frac{6\sin^{-1}x}{\sqrt{1-x^2}}$

$=\sin^{-1}\left(\lim_{x\to\infty}\frac{\sqrt{3}+x}{2+2x}\right)$

$=\sin^{-1}\left(\frac{1}{2}\right)$

$=\frac{\pi}{6}$

A ___ 15. Determine if $\lim_{x\to\infty}\sin^{-1}\left(\frac{\sqrt{3}+x}{2+2x}\right)$ exists, and if it does, find its value.

(A) $\frac{\pi}{6}$ (B) DNE (C) 0 (D) $\frac{\pi}{3}$ (E) $\frac{\pi}{4}$

D ___ 16. Determine if $\lim_{x\to 0}\sin^{-1}\left(\frac{\sqrt{3}+x}{2+2x}\right)$ exists, and if it does, find its value.

(A) $\frac{\pi}{6}$ (B) DNE (C) 0 (D) $\frac{\pi}{3}$ (E) $\frac{\pi}{4}$

$=\sin^{-1}\left(\lim_{x\to 0}\frac{\sqrt{3}+x}{2+2x}\right)=\sin^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{3}$

E 17. Let f be a twice-differentiable function and let g be its inverse. Consider the following equations:

I. $g(f(x))=x$, $f(g(x))=x$

II. $f''(g(x))(g'(x))^2+f'(g(x))g''(x)=0$

III. $g'(x)=\dfrac{1}{f'(g(x))}$

Which of the three above do both f and g satisfy?

(A) I and III only (B) I only (C) III only (D) I and III only (E) I, II, and III

$f(g(x))=x$ Use this to check II.

$\frac{d}{dx}[f'(g(x))\cdot g'(x)]=\frac{d}{dx}[x]$

$\frac{d}{dx}[f'(g(x))\cdot g'(x)]=\frac{d}{dx}[1]$

$f''(g(x))\cdot g'(x)\cdot g'(x)+f'(g(x))\cdot g''(x)=0$

$f''(g(x))\cdot(g'(x))^2+f'(g(x))\cdot g''(x)=0$

A 18. Find the value of $g'(1)$ when g is the inverse of the function $f(x)=2\sin x$, $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$.

(A) $\frac{1}{\sqrt{3}}$ (B) -1 (C) $\frac{1}{\sqrt{2}}$ (D) $-\frac{1}{\sqrt{3}}$ (E) 1

$g(x,y)$

$f(y,x)$

$g(1,\frac{\pi}{6})$

$f(\frac{\pi}{6},1)$

$f(x)=1=2\sin x$

$\sin x=\frac{1}{2}$

$x=\frac{\pi}{6}$

$f'(x)=2\cos x$

$f'(\frac{\pi}{6})=2\cos\frac{\pi}{6}=\sqrt{3}$

$g'(1)=\frac{1}{\sqrt{3}}$

D 19. Suppose g is the inverse function of a differentiable function f and $G(x)=\dfrac{1}{g(x)}$. If $f(3)=7$ and $f'(3)=\frac{1}{9}$, find $G'(7)$.

$f(3,7)$

$g(7,3)$

$f'(3)=\frac{1}{g'(7)}=\frac{1}{9}$

(A) -5 (B) 4 (C) 6 (D) -1 (E) -4

$G'(x)=-1\,g(x)^{-2}\cdot g'(x)$

$G'(x)=\frac{-g'(x)}{g^2(x)}$ $G'(7)=\frac{-g'(7)}{g^2(7)}=\frac{-9}{(3)^2}=-1$

A 20. Find $\dfrac{dy}{dx}$ when $\tan(2x-y)=2x$ (Preview your answer choices first. Notice there is no trig!)

Since no trig should solve for y first!

(A) $\dfrac{dy}{dx}=\dfrac{8x^2}{1+4x^2}$

(B) $\dfrac{dy}{dx}=-\dfrac{8x^2}{1+4x^2}$

(C) $\dfrac{dy}{dx}=-\dfrac{4y^2}{2+x^2}$

(D) $\dfrac{dy}{dx}=\dfrac{4y^2}{2+x^2}$

(E) $\dfrac{dy}{dx}=-\dfrac{8x^2}{1+4y^2}$

$\tan^{-1}[\tan(2x-y)]=\tan^{-1}(2x)$

$2x-y=\tan^{-1}2x$

$y=2x-\tan^{-1}2x$

$\frac{d}{dx}[y]=\frac{d}{dx}[2x-\tan^{-1}2x]$

$\frac{dy}{dx}=2-\left(\frac{1}{1+(2x)^2}\right)(2)$

$\frac{dy}{dx}=2\left(\frac{1+4x^2}{1+4x^2}\right)-\frac{2}{1+4x^2}$

$\frac{dy}{dx}=\frac{8x^2}{1+4x^2}$