Name KEY Date________ Period______

Worksheet 5.4—Concavity and the Second Derivative Test

Show all work. No calculator unless otherwise stated.

Multiple Choice

A

1. If $a < 0$, the graph of $y = ax^3 + 3x^2 + 4x + 5$ is concave up on

(A) $\left(-\infty, -\frac{1}{a}\right)$ (B) $\left(-\infty, \frac{1}{a}\right)$ (C) $\left(-\frac{1}{a}, \infty\right)$ (D) $\left(\frac{1}{a}, \infty\right)$ (E) $(-\infty, -1)$

$y' = 3ax^2 + 6x + 4$

$y'' = 6ax + 6 = 0$

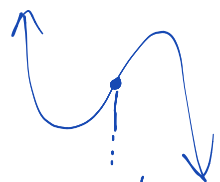
$6ax = -6$

$x = \frac{-6}{6a}$

$\boxed{x = -\frac{1}{a}}$ p.i.v.

$a<0$ so x is positive

since $a < 0$
the graph of y comes in the top & exits the bottom



$x = -\frac{1}{a}$

y is concave up on $\left(-\infty, -\frac{1}{a}\right)$

y is concave down on $\left(-\frac{1}{a}, \infty\right)$

E

2. If $f(0) = f'(0) = f''(0) = 0$, which of the following must be true about the graph of *f*?

(A) There is a local max at the origin (B) There is a local min at the origin
(C) There is no local extremum at the origin (D) There is a point of inflection at the origin
(E) There is a horizontal tangent at the origin

$f(0) = 0$ (x-intercept/y-int at origin)

$f'(0) = 0$ (Horz tangent at origin)

$f''(0) = 0$ possible inflection value at $x = 0$

C

3. The *x*-coordinates of the points of inflection of the graph of $y = x^5 - 5x^4 + 3x + 7$ are

(A) 0 only (B) 1 only (C) 3 only (D) 0 and 3 (E) 0 and 1

$y' = 5x^4 - 20x^3 + 3$

$y'' = 20x^3 - 60x^2$

$y'' = 20x^2(x-3) = 0$

$x = 0, x = 3$
possible inflection values (p.i.v.'s)

x	-1	0	1	3	4
y''	$-$		$-$		$+$

y has an inflection value at $x=3$
since y'' changes from neg to pos at $x=3$

4. Which of the following conditions would enable you to conclude that the graph of f has a point of inflection at $x = c$?

(A) There is a local max of f' at $x = c$ (B) $f''(c) = 0$ (C) $f''(c)$ does not exist

only means f has a p.i.v. at x=c, not an i.v.

(D) The sign of f' changes at $x = c$ (E) f is a cubic polynomial and $c = 0$

* the local extrema of f'(x) are the inflection values of f(x).

* Since f' has a local max @ x=c, its derivative, namely f'', changes from pos to neg at x=c.

B

5. Let f be a twice-differentiable function on $(-\infty, \infty)$ such that the equation $f(x) = 0$ has exactly 3 real roots, all distinct. Consider the following possibilities:

f and f' are continuous

could be of multiplicities greater than 1!

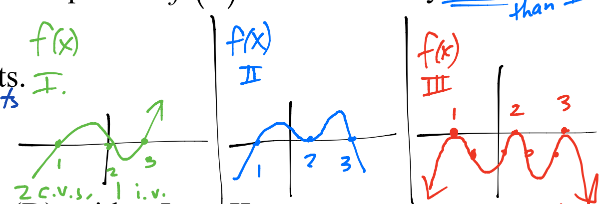
I. the equation $f'(x) = 0$ ~~has at most 3 distinct roots~~.

no more than 3 horiz. tangents

II. the equation $f''(x) = 0$ has at least 1 root.

Which of these properties will f have?

(A) I only (B) II only (C) I and II (D) neither I nor II



F

6. If f is a continuous function on $(-5, 3)$ whose graph is given at right, which of the following properties are satisfied?

I. $f''(x) > 0$ on $(-2, 1)$, False, cc down on (-1,1)

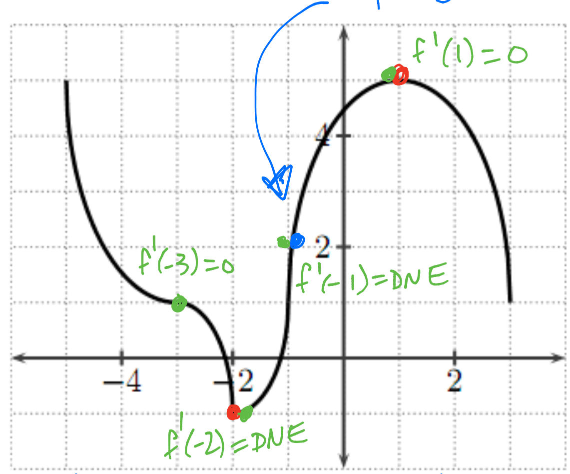
concave up

II. f has exactly 2 local extrema, True (in Red)

III. f has exactly 4 critical points, True, in Green

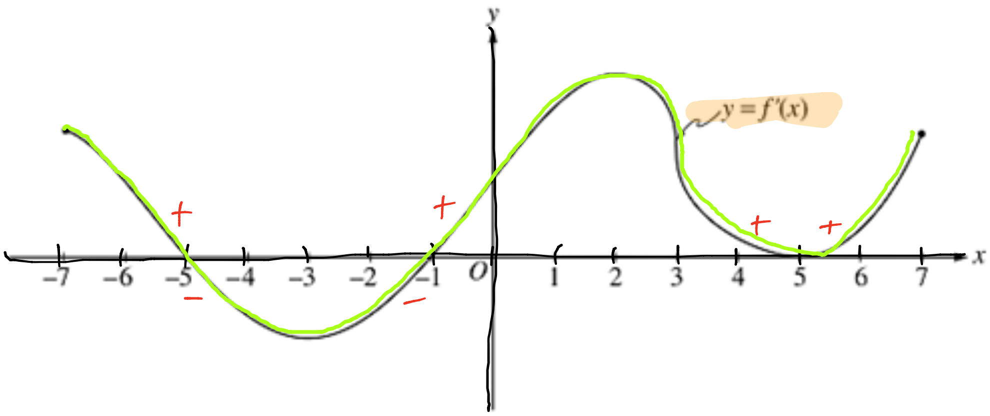
(A) I, II, and III (B) II only (C) I and III (D) I and II
(E) III only (F) II and III (G) I only

vertical tangent



f' is undefined at x=-2 (cusp) and at x=-1 (vert tangent), So f'' is undefined here, too!

Free Response



7. The figure above shows the graph of f', the derivative of the function f, for $-7 \le x \le 7$. The graph of f' has horizontal tangent lines at $x = -3$, $x = 2$, and $x = 5$, and a vertical tangent line at $x = 3$.

(a) Find all the values of x, for $-7 < x < 7$, at which f attains a relative minimum. Justify your answer.

$f' = 0$ at $x = -5, -1, 5$
$f' = \text{DNE}$ at $x = 3$

* f has a relative min at $x = -1$, since $f'(x)$ changes from neg to pos at $x = -1$.

(b) Find all the values of x, for $-7 < x < 7$, at which f attains a relative maximum. Justify your answer.

$f' = 0$ at $x = -5, -1, 5$
$f' = \text{DNE}$ at $x = 3$

* f has a relative max at $x = -5$, since $f'(x)$ changes from pos to neg at $x = -5$

(c) Find all the values of x, for $-7 < x < 7$, at which $f''(x) < 0$.

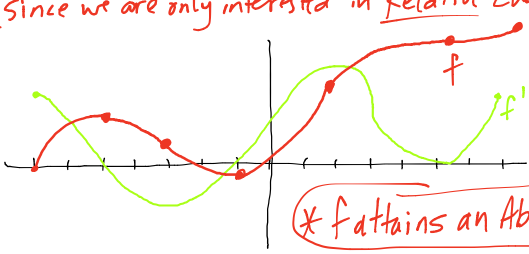
$f'' < 0$ when the slopes of $f'(x)$ are defined and negative, that is, when the graph of f' is decreasing

This happens on the intervals $\boxed{(-7,-3) \cup (2,3) \cup (3,5)}$

since f' has a vert. tang. line at $x=3$, $f''(3) = \text{DNE}$, so we must step over it.

(d) At what value of x, for $-7 < x < 7$, does f attain its absolute maximum? Justify your answer.

For now, let's answer this question by sketching a possible graph of $f(x)$ assuming that $f(-7) = 0$ (since we are only interested in Relative Change in y-values from our starting value.



$\boxed{\text{* } f \text{ attains an Absolute max of } f(7) \text{ at } x = 7.}$

8. Let h be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of h is given by

$$h'(x) = \frac{x^2 - 2}{x}, \ \forall x \neq 0.$$

(a) Find all values of x for which the graph of h has a horizontal tangent, and determine whether h has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

$h'(x) = 0$

$x^2 - 2 = 0$

$\boxed{x = -\sqrt{2}, x = \sqrt{2}}$

Horz. tangs.

$h'(x) = DNE$

$\boxed{x = 0}$

Not a c.v. but it is a discon and goes in the chart

x	-2	$-\sqrt{2}$	-1	0	1	$\sqrt{2}$	2
h'	$-$		$+$		$-$		$+$

*h has relative minimums at $x = -\sqrt{2}$ and $x = \sqrt{2}$ since h' changes from negative to positive at both $x = -\sqrt{2}$ and $x = \sqrt{2}$.

(b) On what intervals, if any, is the graph of h concave up? Justify your answer.

$h' = \frac{x^2 - 2}{x}$

$h'' = \frac{(x)(2x) - (x^2 - 2)}{x^2}$

$h'' = \frac{2x^2 - x^2 + 2}{x^2}$

$\boxed{h'' = \frac{x^2 + 2}{x^2} > 0 \ \forall x \neq 0} \rightarrow$ no inflection values

So $h(x)$ is concave up on $(-\infty, 0) \cup (0, \infty)$

(c) Write an equation for the tangent to the graph of h at $x = 4$.

$h(4) = \boxed{-3}$

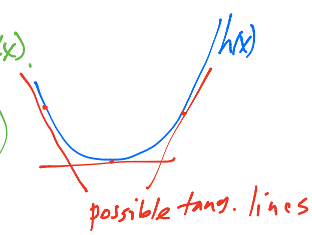
$h'(4) = \frac{4^2 - 2}{4} = \frac{14}{4} = \boxed{\frac{7}{2}}$

eq: $\boxed{y = -3 + \frac{7}{2}(x - 4)}$

(d) Does the line tangent to the graph of h at $x = 4$ lie above or below the graph of h for $x > 4$? Why?

Since $h(x)$ is concave up for all $x \neq 0$, the tangent lines are Below the curve of $h(x)$.

ex)



9. A cubic polynomial function f is defined by $f(x) = 4x^3 + ax^2 + bx + k$, where a, b, and k are constants. The function f has a local minimum at $x = -1$, and the graph of f has a point of inflection at $x = -2$. Find the values of a and b.

$f'(-1) = 0$

$f''(-2) = 0$

$f'(x) = 12x^2 + 2ax + b, \ f''(x) = 24x + 2a$

$f'(-1) = 0 = 12 - 2a + b$

$\boxed{2a - b = 12}$

$f''(-2) = 0 = -48 + 2a$

$2a = 48$

$\boxed{a = 24}$

So $2a - b = 12$

becomes $2(24) - b = 12$

$b = 48 - 12$

$\boxed{b = 36}$

10. For each of the following, i) find the open intervals on which f is increasing or decreasing, ii) find the local max and min values of f, and iii) find the intervals of concavity and inflection points.

(a) $f(x)=\dfrac{x^2}{x^2+3}$

No Discons

$f'(x)=\dfrac{(x^2+3)(2x)-(x^2)(2x)}{(x^2+3)^2}$

$f'(x)=\dfrac{2x[x^2+3-x^2]}{(x^2+3)^2}$

$f'(x)=\dfrac{6x}{(x^2+3)^2}$, $f'(x)=0$ $\boxed{x=0}$ c.v.

x	-1	0	1
f'	−		+

f is inc on $(0,\infty)$ and dec on $(-\infty,0)$.
f has a rel min at $x=0$ since f' changes from neg to pos at $x=0$.

$f''(x)=\dfrac{(x^2+3)^2(6)-(6x)(2)(x^2+3)(2x)}{(x^2+3)^4}$

$f''(x)=\dfrac{6(x^2+3)[x^2+3-4x^2]}{(x^2+3)^4}$

$f''(x)=\dfrac{6(x^2+3)(3-3x^2)}{(x^2+3)^4}$

$f''(x)=\dfrac{6(3)(1-x^2)}{(x^2+3)^3}$

$f''(x)=\dfrac{18(1-x)(1+x)}{(x^2+3)^3}$

$f''(x)=0$
$\boxed{x=1,-1}$ p.i.v.s

x	-2	-1	0	1	2
f''	−		+		−

f is concave up on $(-1,1)$ & concave down on $(-\infty,-1)\cup(1,\infty)$
f has inflection values at $x=-1$, since f'' changes from neg to pos at $x=-1$ and at $x=1$ since f'' changes from pos to neg at $x=1$.

(b) $f(x)=\cos^2 x-2\sin x,\ 0\le x\le 2\pi$

No Discons!

$f(x)=(\cos x)^2-2\sin x$
$f'(x)=2(\cos x)(-\sin x)-2\cos x$
$f'(x)=-2\sin x\cos x-2\cos x$
$f'(x)=-2\cos x(\sin x+1)=0$

$f'(x)=0$: $\cos x=0$, $\sin x=-1$
$x=\frac{\pi}{2},\frac{3\pi}{2}$, $x=\frac{3\pi}{2}$
So f has critical values at $\boxed{x=\frac{\pi}{2},\frac{3\pi}{2}}$

x	π/6	π/2	π	3π/2	11π/6
f'	−		+		−

f is decreasing on $(0,\frac{\pi}{2})\cup(\frac{3\pi}{2},2\pi)$ and increasing on $(\frac{\pi}{2},\frac{3\pi}{2})$.
f has a relative minimum at $x=\frac{\pi}{2}$ since f' changes from negative to pos. at $x=\frac{\pi}{2}$.
f has a relative maximum at $x=\frac{3\pi}{2}$ since f' changes from pos. to neg. at $x=\frac{3\pi}{2}$.

$f''(x)=(2\sin x)(\sin x+1)-2\cos x(\cos x)$
$f''(x)=2\sin^2 x+2\sin x-2(\cos^2 x)$
$f''(x)=2\sin^2 x+2\sin x-2(1-\sin^2 x)$
$f''(x)=2\sin^2 x+2\sin x-2+2\sin^2 x$
$f''(x)=4\sin^2 x+2\sin x-2$
$f''(x)=2(2\sin^2 x+\sin x-1)=0$
$f''(x)=2(2\sin x-1)(\sin x+1)=0$
$2\sin x-1=0$ or $\sin x+1=0$
$\sin x=\frac{1}{2}$ or $\sin x=-1$
$x=\frac{\pi}{6},\frac{5\pi}{6}$ or $x=\frac{3\pi}{2}$

x	0	π/6	π/2	5π/6	π	3π/2	11π/6
f''	−		+		−		−

*f has inflection values at $x=\frac{\pi}{6}$ & $\frac{5\pi}{6}$

f is concave down on $(0,\frac{\pi}{6})\cup(\frac{5\pi}{6},\frac{3\pi}{2})$ and concave up on $(\frac{\pi}{6},\frac{5\pi}{6})$.

(c) $f(x)=\dfrac{\ln x}{\sqrt{x}}$

Domain: $\boxed{x>0}$

$f(x)=\dfrac{\ln x}{x^{1/2}}$, $f'(x)=\dfrac{(x^{1/2})(\frac{1}{x})-(\ln x)(\frac{1}{2}x^{-1/2})}{(x^{1/2})^2}\cdot\dfrac{(2x)}{(2x)}=\dfrac{2x^{1/2}-x^{1/2}\ln x}{2x^2}$

$f'(x)=\dfrac{x^{1/2}(2-\ln x)}{2x^2}=\dfrac{\sqrt{x}(2-\ln x)}{2x^2}$, $f'=0$ when $\sqrt{x}=0$, $\boxed{x=0}$ Not in domain; $2-\ln x=0$, $\ln x=2$, $\boxed{x=e^2}$ c.v.

x	e	e²	e³
f'	+		−

*f is inc on $(0,e^2)$ & dec on (e^2,∞)
*f has a relative max @ $x=e^2$ since f' changes from pos to neg at $x=e^2$.

$f'(x)=\dfrac{x^{1/2}(2-\ln x)}{2x^2}=\dfrac{2-\ln x}{2x^{3/2}}$ ← better $f'(x)$

$f''(x)=\dfrac{2x^{3/2}(-\frac{1}{x})-(2-\ln x)(3x^{1/2})}{(2x^{3/2})^2}=\dfrac{-2x^{1/2}-6x^{1/2}+3x^{1/2}\ln x}{4x^3}$

$f''(x)=\dfrac{-8x^{1/2}+3x^{1/2}\ln x}{4x^3}=\dfrac{x^{1/2}[3\ln x-8]}{4x^3}=\dfrac{3\ln x-8}{4x^{5/2}}=f''$

$f''=0$ when $3\ln x-8=0$
$\ln x=\frac{8}{3}$
$\boxed{x=e^{8/3}}$ p.i.v.

$f''=$ DNE when $4x^{5/2}=0$
$\boxed{x=0}$ not in domain

x	e^{7/3}	e^{8/3}	e³
f''	−		+

* f is concave down on $(0,e^{8/3})$ and concave up on $(e^{8/3},\infty)$.
* f has an inflection value at $x=e^{8/3}$ since f'' changes from neg to pos at $x=e^{8/3}$

11. Find the local max and min values of $f(x)=x+\sqrt{1-x}$ using both the First and Second Derivative Tests. Which method do you prefer?

Domain: $1-x\ge 0$ $\boxed{x\le 1}$, $f(x)=x+(1-x)^{1/2}$

$f'(x)=1+\frac{1}{2}(1-x)^{-1/2}(-1)$

$f'(x)=1-\dfrac{1}{2\sqrt{1-x}}$

$f'(x)=\dfrac{2\sqrt{1-x}-1}{2\sqrt{1-x}}$

$f'(x)=0$
$2\sqrt{1-x}-1=0$
$2\sqrt{1-x}=1$
$\sqrt{1-x}=\frac{1}{2}$
$1-x=\frac{1}{4}$
$x=1-\frac{1}{4}$
$\boxed{x=\frac{3}{4}}$ c.v.

x	1/4	3/4	1
f'	+		

1st Deriv Test: By the 1st Deriv Test, since f' changes from pos to neg at $x=3/4$, f has a Relative Max at $x=3/4$.

$f''(x)=\left(-\frac{1}{4}\right)(1-x)^{-3/2}(-1)(-1)$

$f''(x)=\dfrac{-1}{4\sqrt{(1-x)^3}}$

2nd Deriv Test: $f''\left(\frac{3}{4}\right)=\dfrac{-1}{4\left(\frac{1}{2}\right)^3}<0$

so f is concave down at $x=3/4$
so f has a Rel. Max at $x=3/4$

12. Sketch the graph of a function that satisfies all of the following conditions.
(a) $f'(x) > 0$ for all $x \neq 1$, VA at $x = 1$, $f''(x) > 0$ if $x < 1$ or $x > 3$, and $f''(x) < 0$ if $1 < x < 3$.

(b) $f'(x) > 0$ if $|x| < 2$, $f'(x) < 0$ if $|x| > 2$, $f'(2) = 0$, $\lim_{x \to \infty} f(x) = 1$, $f(-x) = -f(x)$, $f''(x) < 0$ if $0 < x < 3$, and $f''(x) > 0$ if $x > 3$

13. Coffee is being poured into Mr. Korpi's mug (shown below) at a constant rate (measured in volume per unit of time). Sketch a rough graph of the depth of the coffee in the mug as a function of time. Account for the shape of the graph in terms of concavity. What is the significance of the inflection point?