Name KEY Date________ Period______

Worksheet 7.3—Separable Differential Equations
Show all work. No Calculator unless specified.

Multiple Choice

C 1. (OK, so you can use your calculator right away on a non-calculator worksheet. Use it on this one.)A sample of Kk-1234 (an isotope of Kulmakorpium) loses 99% of its radioactive matter in 199 hours. What is the half-life of Kk-1234?

(A) 4 hours (B) 6 hours (C) 30 hours (D) 100.5 hours (E) 143 hours

$y = Ce^{kt}$ $(0, 100)$ ① $(199, 1)$ ②

① $y = 100e^{kt}$

② $1 = 100e^{199k}$

$\frac{1}{100} = e^{199k}$

$\ln\left(\frac{1}{100}\right) = 199k$

$k = \frac{\ln(1/100)}{199}$

$\boxed{k = \frac{-\ln(100)}{199}}$

So $y = 100e^{\left(-\frac{\ln 100}{199}t\right)}$

For half-life, let $y = \frac{100}{2} = 50$

$50 = 100e^{\left(-\frac{\ln 100}{199}\right)t}$

$\frac{1}{2} = e^{-\frac{\ln(100)}{199}t}$

$\ln\left(\frac{1}{2}\right) = -\frac{\ln(100)}{199}t$

$\frac{199\ln\left(\frac{1}{2}\right)}{-\ln(100)} = t$

$t = \frac{-199\ln(2)}{-\ln(100)}$

$\boxed{t = \frac{199\ln 2}{\ln 100} \text{ yrs}} \approx \boxed{29.952 \text{ yrs}}$

E 2. In which of the following models is $\frac{dy}{dt}$ directly proportional to y?

I. $y = e^{kt} + C$

II. $y = Ce^{kt}$ ✓

III. $y = 28^{kt}$ ✓

IV. $y = 3\left(\frac{1}{2}\right)^{3t+1}$ ✓

$y = Ce^{kt}$

or

$y = \pm Ae^{(kt+c)}$

or

$y = A \cdot b^{(kt+c)}$

or

$y = A \cdot b^{kt}$

(A) I only (B) II only (C) I and II only (D) II and III only (E) II, III, and IV (F) all of them

C

3. (Use your calculator on this one, too, but get the exact answer first.) The rate at which acreage is being consumed by a plot of kudzu is proportional to the number of acres already consumed at time *t*. If there are 2 acres consumed when $t=1$ and 3 acres consumed when $t=5$, how many acres will be consumed when $t=8$?

(A) 3.750 (B) 4.000 (C) 4.066 (D) 4.132 (E) 4.600

$\frac{dy}{dt}=ky$ (1,2) (5,3)

so $y=Ce^{kt}$

① $2=Ce^{k}$

② $3=Ce^{5k}$

②÷①: $\frac{3}{2}=e^{4k}$

$\ln(3/2)=4k$

$\boxed{k=\frac{\ln(1.5)}{4}}$

using eq ①

$2=Ce^{\frac{\ln 1.5}{4}}$

$2=Ce^{\ln(1.5)^{1/4}}$

$2=C\cdot(1.5)^{1/4}$

$C=\frac{2}{\sqrt[4]{1.5}}$

So $y=\frac{2}{\sqrt[4]{1.5}}e^{\left(\frac{\ln(1.5)}{4}\right)t}$

$y(8)=\frac{2}{\sqrt[4]{1.5}}e^{\left(\frac{\ln 1.5}{4}\right)8}$

$\boxed{y(8)=4.066209}$

Free Response

For problems 4 – 13, find the general solution to the following differential equations, then find the particular solution using the initial condition.

4. $\frac{dy}{dx}=\frac{x}{y},\ y(1)=-2$

$\int y\,dy=\int x\,dx$

$\frac{1}{2}y^2=\frac{1}{2}x^2+C$

$y^2=x^2+C \quad (2C=C)$

gen soln $\boxed{y=\pm\sqrt{x^2+C}}$

At (1,-2):

$-2=\pm\sqrt{1+C}$

$4=1+C$

$\boxed{C=3}$

So $\boxed{y=-\sqrt{x^2+3}}$

negative since $y=-2$ at $x=1$

5. $\frac{dy}{dx}=-\frac{x}{y},\ y(4)=3$

$\int y\,dy=\int -x\,dx$

$\frac{1}{2}y^2=-\frac{1}{2}x^2+C$

$y^2=-x^2+C \quad (2C=C)$

gen solution $\boxed{y=\pm\sqrt{-x^2+C}}$

At (4,3):

$3=\pm\sqrt{-(4^2)+C}$

$3=\pm\sqrt{-16+C}$

$9=-16+C$

$\boxed{C=25}$

So $y=+\sqrt{-x^2+25}$

$\boxed{y=\sqrt{25-x^2}}$

6. $\frac{dy}{dx}=\frac{y}{x},\ y(2)=2$

$\int\frac{1}{y}\,dy=\int\frac{1}{x}\,dx$

$\ln|y|=\ln|x|+C$

$|y|=e^{\ln|x|+C}$

$|y|=e^{\ln|x|}\cdot e^{C}, \ (e^C=C)$

gen soln $\boxed{y=C|x|}$

At (2,2):

$2=C|2|$

$2=2C$

$\boxed{C=1}$

So $\boxed{y=|x|}$

7. $\frac{dy}{dx}=2xy,\ y(0)=-3$

$\int\frac{1}{y}\,dy=\int 2x\,dx$

$\ln|y|=x^2+C$

$|y|=e^{x^2+C}$

gen soln $\boxed{y=Ce^{x^2}}$

At (0,-3):

$-3=Ce^0$

$\boxed{C=-3}$

So $\boxed{y=-3e^{x^2}}$

8. $\frac{dy}{dx}=(y+5)(x+2),\ y(0)=-1$

$\int\frac{1}{y+5}\,dy=\int(x+2)\,dx$

$\ln|y+5|=\frac{1}{2}x^2+2x+C$

$|y+5|=e^{\frac{1}{2}x^2+2x+C}$

$|y+5|=e^{\frac{1}{2}x^2+2x}\cdot e^{C}$

$y+5=Ce^{\frac{1}{2}x^2+2x}$

gen soln $\boxed{y=Ce^{\left(\frac{1}{2}x^2+2x\right)}-5}$

At (0,-1):

$-1=Ce^0-5$

$\boxed{C=4}$

So $\boxed{y=4e^{\left(\frac{1}{2}x^2+2x\right)}-5}$

9. $\frac{dy}{dx}=\cos^2 y,\ y(0)=0$

$\frac{1}{\cos^2 y}dy=dx$

$\int\sec^2 y\,dy=\int 1\,dx$

$\tan y=x+C$

gen soln $\boxed{y=\tan^{-1}(x+C)}$

At (0,0):

$0=\tan^{-1}(C)$

$C=\tan 0$

$\boxed{C=0}$

So $\boxed{y=\tan^{-1}x}$

10. $\dfrac{dy}{dx}=(\cos x)e^{y+\sin x},\ y(0)=0$

$\frac{dy}{dx}=\cos x\cdot e^{y}\cdot e^{\sin x}$

$\int e^{-y}dy=\int \cos x\cdot e^{\sin x}dx$

$-e^{-y}=e^{\sin x}+c$

$e^{-y}=-e^{\sin x}+c \quad (-c=c)$

$-y=\ln(c-e^{\sin x})$

gen soln $\boxed{y=-\ln(c-e^{\sin x})}$

At $(0,0)$:

$0=-\ln(c-e^{0})$

$0=\ln(c-1)$

$e^{0}=c-1$

$c=1+1$

$\boxed{c=2}$

So $\boxed{y=-\ln(2-e^{\sin x})}$

11. $\dfrac{dy}{dx}=e^{x-y},\ y(0)=2$

$\frac{dy}{dx}=e^{x}\cdot e^{-y}$

$\int e^{y}dy=\int e^{x}dx$

$e^{y}=e^{x}+c$

gen soln $\boxed{y=\ln(e^{x}+c)}$

At $(0,2)$:

$2=\ln(1+c)$

$e^{2}=1+c$

$\boxed{c=e^{2}-1}$

So $y=\ln(e^{2}+e^{2}-1)$

$\boxed{y=\ln(2e^{2}-1)}$

12. $\dfrac{dy}{dx}=-2xy^2,\ y(1)=0.25$

$\int y^{-2}dy=\int -2x\,dx$

$-y^{-1}=-x^{2}+C$

$\frac{1}{y}=x^{2}+c \quad (-c=c)$

$\boxed{y=\frac{1}{x^{2}+c}}$

At $(1,\frac{1}{4})$:

$\frac{1}{4}=\frac{1}{1+c}$

$4=1+c$

$\boxed{c=3}$

So $\boxed{y=\frac{1}{x^{2}+3}}$

13. $\dfrac{dy}{dx}=\dfrac{4\sqrt{y}\ln x}{x},\ y(e)=1$

$\int y^{-1/2}dy=\int 4\left(\frac{1}{x}\right)\ln x\,dx$

$2y^{1/2}=4\left(\frac{1}{2}\right)(\ln x)^{2}+C$

$2\sqrt{y}=2\ln^{2}x+C$

$\sqrt{y}=\ln^{2}x+C \quad * \frac{C}{2}=C$

$y=(\ln^{2}x+C)^{2}$ (gen solution)

for $y(e)=1$: $1=((\ln e)^{2}+c)^{2}$

$1=(1+c)$

$c=0$

So, $y=(\ln^{2}x)^{2}$

or $y=\ln^{4}x$

For problems 14 – 17, find the solution of the differential equation $\dfrac{dy}{dt}=ky$ that satisfies the given conditions.

14. $k=1.5,\ y(0)=100$

$y=Ce^{kt}$

$\boxed{y=100e^{1.5t}}$

15. $k=-0.5,\ y(0)=200$

$y=Ce^{kt}$

$\boxed{y=200e^{-\frac{1}{2}t}}$

16. $y(0)=50,\ y(5)=100$

$y=Ce^{kt}$

$100=50e^{5k}$

$2=e^{5k}$

$\ln 2=5k$

$\boxed{k=\frac{\ln 2}{5}}$

$\boxed{\text{So } y=50e^{(\frac{1}{5}\ln 2)t}}$

or $y=50(2^{t/5})$

17. $y(1)=55,\ y(10)=30$ (divide one by the other)

$y=Ce^{kt}$

① $55=Ce^{k}$

② $30=Ce^{10k}$

②÷①: $\frac{30}{55}=e^{9k}$

$\ln\left(\frac{30}{55}\right)=9k$

$\boxed{k=\frac{1}{9}\ln\left(\frac{6}{11}\right)}$

So $55=Ce^{\frac{1}{9}\ln\left(\frac{6}{11}\right)}$

$55=C\left(\frac{6}{11}\right)^{1/9}$

$C=55\cdot\left(\frac{6}{11}\right)^{-1/9}$

So $\boxed{y=55\left(\frac{6}{11}\right)^{-1/9}\cdot e^{\left(\frac{1}{9}\ln\left(\frac{6}{11}\right)\right)t}}$

or

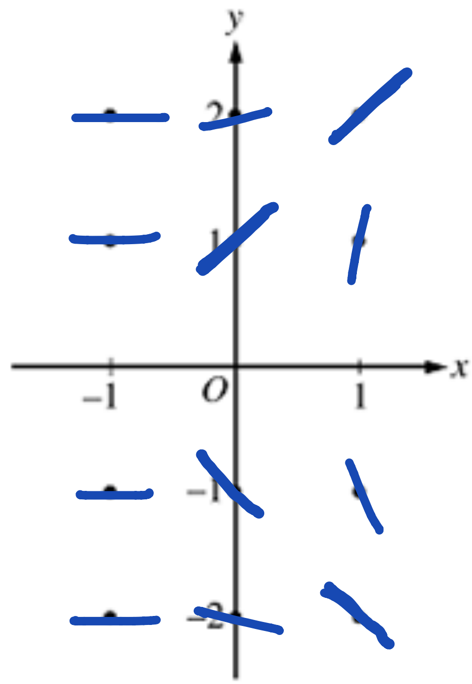
$y=55\left(\frac{6}{11}\right)^{-1/9}\cdot\left(\frac{6}{11}\right)^{t/9}$

$y=55\left(\frac{6}{11}\right)^{(t-1)/9}$

18. AP 2010B-5 (No Calculator)

Consider the differential equation $\frac{dy}{dx} = \frac{x+1}{y}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and for $-1 < x < 1$, sketch the solution curve that passes through the point $(0,-1)$.



(b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy-plane for which $y \neq 0$. Describe all points in the xy-plane, $y \neq 0$, for which $\frac{dy}{dx} = -1$.

$\frac{x+1}{y} = -1$

$x+1 = -y$

$\boxed{y = -x-1}$

so $\frac{dy}{dx} = -1$ for all pts $y \neq 0$ on the line $y = -x-1$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -2$.

$\frac{dy}{dx} = \frac{x+1}{y}$

$\int y\,dy = \int (x+1)\,dx$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 + x + C$

$y^2 = x^2 + 2x + C$ $(2C = C)$

gen soln $\boxed{y = \pm\sqrt{x^2+2x+C}}$

At $(0,-2)$:

$-2 = \pm\sqrt{C}$

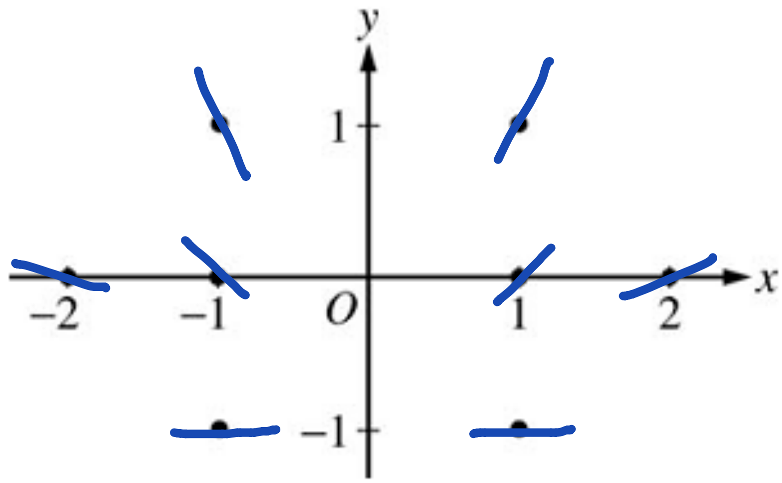
$\boxed{C=4}$

So $\boxed{y = -\sqrt{x^2+2x+4}}$ ← negative since $f(0) = -2$

19. AP 2006-5

Consider the differential equation $\frac{dy}{dx} = \frac{1+y}{x}$, where $\boxed{x \neq 0}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the eight points indicated.



(b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(-1) = 1$ and state its domain.

$$\frac{dy}{dx} = \frac{1+y}{x}$$

$$\int \frac{1}{1+y}\,dy = \int \frac{1}{x}\,dx$$

$$\ln|1+y| = \ln|x| + C$$

$$|1+y| = e^{\ln|x|+C}$$

$$1+y = Ce^{\ln|x|}$$

gen soln $\boxed{y = C|x| - 1}$

At $(-1,1)$:

$$1 = C|-1| - 1$$

$$1 = C - 1$$

$$\boxed{C = 2}$$

So $\boxed{y = 2|x| - 1}$

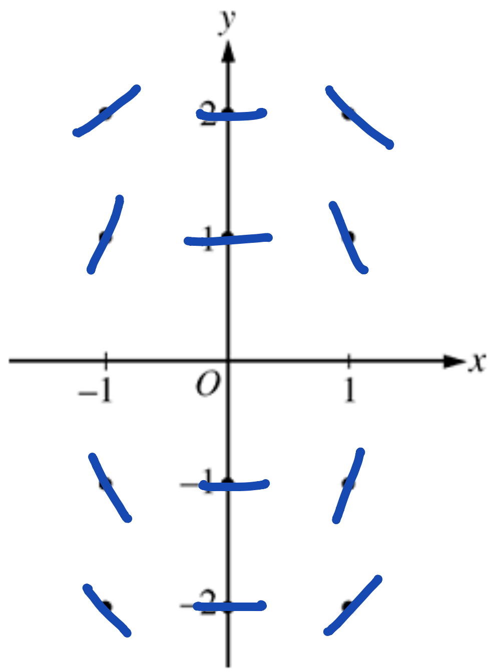
* Since $\frac{dy}{dx}$ has a discontinuity at $x=0$ & question also tells us that $x \neq 0$, which applies to ALL solution curves, too, we only sketch/discuss functions (pass vert line test) that are continuous.

Since our initial condition is $x = -1 < 0$ and since $x \neq 0$, the domain of $y = 2|x| - 1$ is restricted to $\boxed{x < 0}$.

20. AP 2005-6

Consider the differential equation $\frac{dy}{dx} = -\frac{2x}{y}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.



(b) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = -1$. Write an equation for the line tangent to the graph of f at $(1,-1)$ and use it to approximate $f(1.1)$.

$\left.\frac{dy}{dx}\right|_{(1,-1)} = 2$

eq: $\boxed{\mathcal{L}(x) = -1 + 2(x-1)}$

$f(1.1) \approx \mathcal{L}(1.1) = -1 + 2(1.1-1)$

"squiggles"

$= -1 + 2(.1)$

$= -1 + .2$

$= \boxed{-0.8}$

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(1) = -1$.

$\frac{dy}{dx} = \frac{-2x}{y}$

$\int y\,dy = \int -2x\,dx$

$\frac{1}{2}y^2 = -x^2 + C$

$y^2 = -2x^2 + C \quad (2C = C)$

gen soln $\boxed{y = \pm\sqrt{C - 2x^2}}$

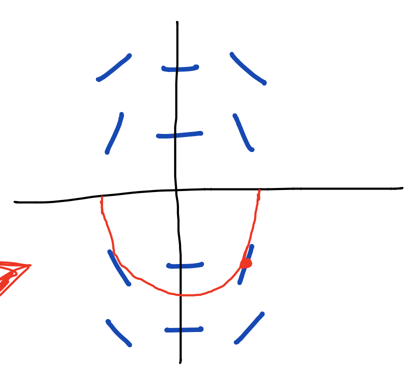
At $(1,-1)$:

$-1 = \pm\sqrt{C-2}$

$1 = C - 2$

$\boxed{C = 3}$

$\boxed{y = -\sqrt{3 - 2x^2}}$ *negative since $f(1) = -1$



* the sketch of this particular solution would only be the BOTTOM portion of the graph below the x-axis (continuous functions remember).