Name KEY Date__________ Period______

Worksheet 5.4—Integration by Parts
Show all work. No calculator unless stated.

Multiple Choice

B 1. If $\int x^2 \cos x\,dx = h(x) - \int 2x\sin x\,dx$, then $h(x) =$

(A) $2\sin x + 2x\cos x + C$ (B) $x^2 \sin x + C$ (C) $2x\cos x - x^2\sin x + C$

(D) $4\cos x - 2x\sin x + C$ (E) $(2-x^2)\cos x - 4\sin x + C$

$u = x^2$, $dv = \cos x\,dx$

$du = 2x\,dx$, $v = \sin x$

$\int x^2 \cos x\,dx = x^2 \sin x - \int 2x\sin x\,dx$

So, $h(x) = x^2 \sin x + C$

B 2. $\int x\sin(5x)dx = I$

(A) $-x\cos(5x) + \sin(5x) + C$ (B) $-\frac{x}{5}\cos(5x) + \frac{1}{25}\sin(5x) + C$ (C) $-\frac{x}{5}\cos(5x) + \frac{1}{5}\sin(5x) + C$

(D) $\frac{x}{5}\cos(5x) + \frac{1}{25}\sin(5x) + C$ (E) $5x\cos(5x) - \sin(5x) + C$

u	dv	+/−
x	$\sin 5x$	+
1	$-\frac{1}{5}\cos 5x$	−
0	$-\frac{1}{25}\sin 5x$	+

$I = -\frac{1}{5}x\cos(5x) - \left(-\frac{1}{25}\right)\sin(5x) + C$

$= -\frac{1}{5}x\cos(5x) + \frac{1}{25}\sin(5x) + C$

C 3. $\int x\csc^2 x\,dx = I$

(A) $\frac{x\csc^3 x}{6} + C$ (B) $x\cot x - \ln|\sin x| + C$ (C) $-x\cot x + \ln|\sin x| + C$

(D) $-x\cot x - \ln|\sin x| + C$ (E) $-x\sec^2 x - \tan x + C$

u	dv	+/−
x	$\csc^2 x$	+
1	$-\cot x$	−
0	$-\ln\|\sin x\|$	+

$I = -x\cot x - (-\ln|\sin x|) + C$

$I = -x\cot x + \ln|\sin x| + C$

C 4. The graph of $y = f(x)$ conforms to the slope field for the differential equation $\frac{dy}{dx} = 4x \ln x$, as shown. Which of the following could be $f(x)$?

$\int dy = \int 4x \ln x \, dx = I$

$y = 2x^2 \ln x - \int \left(\frac{1}{x}\right)(2x^2)\,dx$

$= 2x^2 \ln x - x^2 + C$

(A) $2x^2 \ln^2 x + 3$

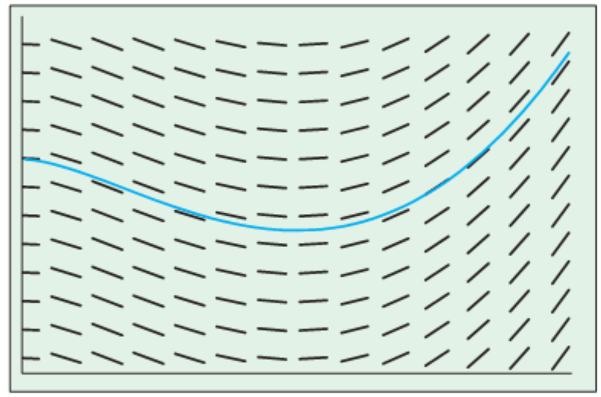
(B) $x^3 \ln x + 3$

(C) $2x^2 \ln x - x^2 + 3$

(D) $(2x^2 + 3)\ln x - 1$

(E) $2x \ln^2 x - \frac{4 \ln^3 x}{3} + 3$

u	dv	+/−
$\ln x$	$4x$	+
$\int \rightarrow \frac{1}{x}$	$2x^2$	−



[0, 2] by [0, 5]

Short Answer

5. Evaluate the following integrals.

(a) $\int xe^{-x} dx = I$

u	dv	+/−
x	e^{-x}	+
1	$-e^{-x}$	−
0	e^{-x}	+

$I = -xe^{-x} - e^{-x} + C$

(b) $\int x^2 \sin(\pi x)\,dx = I$

u	dv	+/−
x^2	$\sin(\pi x)$	+
$2x$	$-\frac{1}{\pi}\cos(\pi x)$	−
2	$-\frac{1}{\pi^2}\sin(\pi x)$	+
0	$\frac{1}{\pi^3}\cos(\pi x)$	−

$I = -\frac{1}{\pi}x^2\cos(\pi x) + \frac{2}{\pi^2}x\sin(\pi x) + \frac{2}{\pi^3}\cos(\pi x) + C$

(c) $\int \sin^{-1} x\,dx = I$

u	dv	+/−
$\sin^{-1} x$	1	+
$\frac{1}{\sqrt{1-x^2}}$	x	−

$I = x\sin^{-1}x - \int x(1-x^2)^{-1/2}dx$

$I = x\sin^{-1}x - \left(-\frac{1}{2}\right)(2)(1-x^2)^{1/2} + C$

$I = x\sin^{-1}x + \sqrt{1-x^2} + C$

(d) $\int \ln^2 x\,dx = I$

$u = (\ln x)^2 \quad dv = dx$

$du = 2(\ln x)\left(\frac{1}{x}\right) \quad v = x$

$I = x\ln^2 x - 2\int \ln x\,dx$

$\int \ln x\,dx = K$

$u = \ln x \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

$K = x\ln x - \int 1\,dx$

$K = x\ln x - x + C$

$I = x\ln^2 x - 2[x\ln x - x] + C$

$I = x\ln^2 x - 2x\ln x + 2 + C$

(e) $\int \arctan 4t\,dt = I$

$u = \arctan 4t \quad dv = dt$

$du = \frac{4}{1+16t^2}dt \quad v = t$

$I = t\arctan 4t - 4\int \frac{t}{1+16t^2}dt$

$I = t\arctan 4t - 4\left(\frac{1}{32}\right)\ln|1+16t^2| + C$

$I = t\arctan 4t - \frac{1}{8}\ln(1+16t^2) + C$

6. Evaluate the following definite integrals. Show the antiderivative. Verify on your calculator.

(a) $\displaystyle\int_0^{\pi} t\sin 3t\,dt = I$

u	dv	+/-
t	$\sin 3t$	+
1	$-\frac{1}{3}\cos 3t$	−
0	$-\frac{1}{9}\sin 3t$	+

$I = -\frac{1}{3}t\cos 3t + \frac{1}{9}\sin 3t\Big|_0^{\pi}$

$= \left(-\frac{\pi}{3}\cos 3\pi + \frac{1}{9}\sin 3\pi\right) - \left(0 + \frac{1}{9}\sin 0\right)$

$= \frac{\pi}{3} + 0 - 0$

$= \frac{\pi}{3} \approx 1.407$

(b) $\displaystyle\int_0^{1}\left(x^2+1\right)e^{-x}\,dx = I$

u	dv	+/−
x^2+1	e^{-x}	+
$2x$	$-e^{-x}$	−
2	e^{-x}	+
0	$-e^{-x}$	−

$I = -(x^2+1)e^{-x} - 2xe^{-x} - 2e^{-x}\Big|_0^1$

$I = \left(-2e^{-1} - 2e^{-1} - 2e^{-1}\right) - \left(-1 - 0 - 2\right)$

$I = -6e^{-1} + 3 \approx 0.792$

(c) $\displaystyle\int_1^{e}\frac{\ln x}{x^2}\,dx = I$

u	dv	+/−
$\ln x$	x^{-2}	+
$\frac{1}{x}$	$-x^{-1}$	−

$I = -\frac{\ln x}{x}\Big|_1^e - \int_1^e\left(\frac{1}{x}\right)\left(-x^{-1}\right)dx$

$I = -\frac{\ln x}{x}\Big|_1^e + \int_1^e x^{-2}\,dx$

$I = -\frac{\ln x}{x} - \frac{1}{x}\Big|_1^e$

$I = \left(-\frac{1}{e} - \frac{1}{e}\right) - \left(0 - 1\right)$

$I = -\frac{2}{e} + 1 \approx 0.264$

(d) $\displaystyle\int_0^{1}\frac{r^3}{\sqrt{4+r^2}}\,dr = I$ (Hint: let $r^3 = r^2\cdot r$)

$I = \int_0^1 r^2\cdot\left(\left(4+r^2\right)^{-1/2}\cdot r\right)dr$

u	dv	+/−
r^2	$(4+r^2)^{-1/2}\cdot r$	+
$2r$	$\frac{1}{\cancel{2}}(\cancel{2})(4+r^2)^{1/2}$	−

$I = r^2(4+r^2)^{1/2}\Big|_0^1 - \int_0^1 (2r)(4+r^2)^{1/2}dr$

$I = r^2\sqrt{4+r^2} - 2\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(4+r^2\right)^{3/2}\Big|_0^1$

$I = r^2\sqrt{4+r^2} - \frac{2}{3}\sqrt{(4+r^2)^3}\Big|_0^1$

$I = \left(\sqrt{5} - \frac{2}{3}\sqrt{125}\right) - \left(0 - \frac{2}{3}(8)\right)$

$I = \sqrt{5} + \frac{16}{3} - \frac{2}{3}\sqrt{125} \approx 0.115$

7. Solve: $\dfrac{dy}{dx} = x\sec^2 x$ and $y = 1$ when $x = 0$.

$\int dy = \int x\sec^2 x\,dx = I$

$y = I$

u	dv	+/−
x	$\sec^2 x$	+
1	$\tan x$	−
0	$-\ln\lvert\cos x\rvert$	+

$I = y = x\tan x + \ln\lvert\cos x\rvert + C$

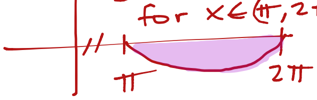
@ $(0,1)$: $1 = 0 + \ln 1 + C$

$C = 1$

So, $y = x\tan x + \ln\lvert\cos x\rvert + 1$

8. Find the area of the region enclosed by the x-axis and the curve $y = x\sin x$ for $\pi \le x \le 2\pi$.

From calculator:
$y = x\sin x < 0$
for $x \in (\pi, 2\pi)$



So, Area $= -\displaystyle\int_{\pi}^{2\pi} x\sin x\,dx = I$

u	dv	+/−
x	$\sin x$	+
1	$-\cos x$	−
0	$-\sin x$	+

Area $= -\left[-x\cos x + \sin x\right]\Big|_{\pi}^{2\pi}$

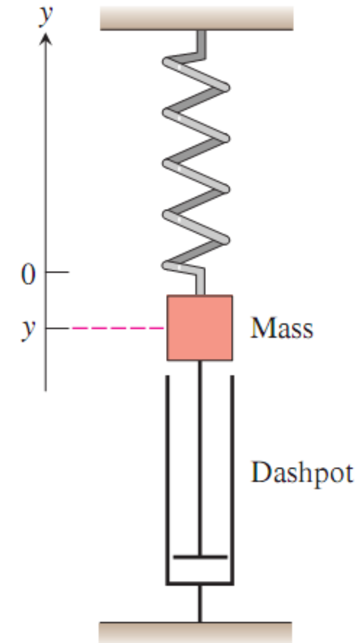
$= x\cos x - \sin x\Big|_{\pi}^{2\pi}$

$= \left(2\pi\cos 2\pi - \sin 2\pi\right) - \left(\pi\cos\pi - \sin\pi\right)$

$= 2\pi - 0 + \pi + 0$

$= 3\pi \approx 9.424$

9. A slowing force, symbolized by the "Dashpot" in the **figure at right**, slows the motion of the weighted spring so that the mass's position at time t is given by $y = 2e^{-t}\cos t$, $t \geq 0$. Find the average position of the mass on the interval $t \in [0, 2\pi]$. Give an exact answer, then verify on your calculator.



This problem is a bit of a wooly booger.
It involves 2 function in the integrand whose derivatives repeat like #13 in the notes.

There is not enough space here, so...
See attached page after example 10.

10. Using u-substitution and then integration by parts, evaluate $\int \sin\sqrt{x}\,dx$.

Let $u = \sqrt{x}$
$x = u^2$
$dx = 2u\,du$

So, $\int \sin\sqrt{x}\,dx$ is now $\int \sin u \cdot 2u\,du$
$= \int 2u \sin u\,du$
$= -2u\cos u + 2\sin u + C$

u	dv	$+/-$
$2u$	$\sin u$	$+$
2	$-\cos u$	$-$
0	$-\sin u$	$+$

So, $\int \sin\sqrt{x}\,dx = -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$

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$$\text{Avg position} = \frac{\int_0^{2\pi}(2e^{-t}\cos t)\,dt}{2\pi - 0}$$

$$\frac{2}{2\pi}\int_0^{2\pi} e^{-t}\cos t\,dt$$

$$\text{Avg position} = \frac{1}{\pi}\int_0^{2\pi} e^{-t}\cos t\,dt = \frac{1}{\pi}\left[e^{-t}\sin t - e^{-t}\cos t + \int_0^{2\pi} -e^{-t}\cos t\,dt\right]$$

$$\frac{1}{\pi}\int_0^{2\pi} e^{-t}\cos t\,dt = \frac{1}{\pi}\left[e^{-t}\sin t - e^{-t}\cos t\right] - \frac{1}{\pi}\int_0^{2\pi} e^{-t}\cos t\,dt$$

$$\frac{1}{\pi}\int_0^{2\pi} e^{-t}\cos t\,dt + \frac{1}{\pi}\int_0^{2\pi} e^{-t}\cos t\,dt = \frac{1}{\pi}\left[e^{-t}\sin t - e^{-t}\cos t\right]$$

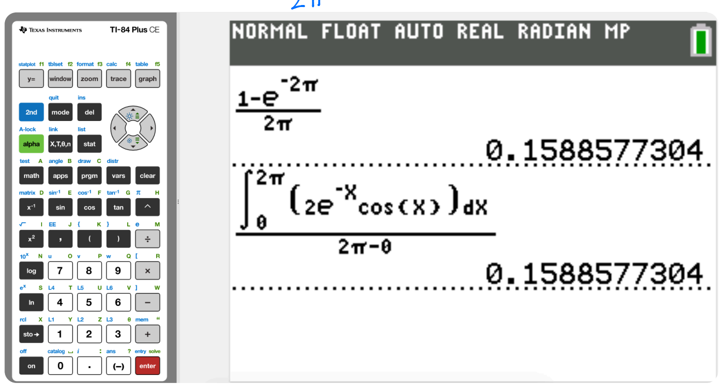
$$\frac{2}{\pi}\int_0^{2\pi} e^{-t}\cos t\,dt = \frac{1}{\pi}\left[e^{-t}\sin t - e^{-t}\cos t\right]\Big|_0^{2\pi}$$

$$\text{Avg position} = \frac{1}{\pi}\int_0^{2\pi} e^{-t}\cos t\,dt = \frac{1}{2\pi}\left[e^{-t}\sin t - e^{-t}\cos t\right]\Big|_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(e^{-2\pi}\sin 2\pi - e^{-2\pi}\cos 2\pi\right) - \left(e^{0}\sin 0 - e^{0}\cos 0\right)\right]$$

$$= \frac{1}{2\pi}\left[\left(0 - e^{-2\pi}\right) - (0 - 1)\right]$$

$$= \frac{1 - e^{-2\pi}}{2\pi} \approx 0.158$$



Integration by Parts TAB METHOD

u	dv	+/-
e^{-t}	$\cos t$	$+$
$-e^{-t}$	$\sin t$	$-$
e^{-t}	$-\cos t$	$+$