Name KEY Date________________________ Period______

Worksheet 5.5—Partial Fractions & Logistic Growth
Show all work. No calculator unless stated.

Multiple Choice

D

1. The spread of a disease through a community can be modeled with the logistic equation $y = \dfrac{600}{1+59e^{-0.1t}}$, where y is the number of people infected after t days. How many people are infected when the disease is spreading the fastest?

(A) 10 (B) 59 (C) 60 (D) 300 (E) 600

Logistic models of the form $y = \dfrac{L}{1+Ce^{-Lkt}}$ grow the fastest when $y = L/2$. In this model, $L = 600$, so $\frac{L}{2} = 300$

B

2. The spread of a disease through a community can be modeled with the logistic equation $y = \dfrac{0.9}{1+45e^{-0.15t}}$, where y is the proportion of people infected after t days. According to the model, what percentage of people in the community will not become infected?

(A) 2% (B) 10% (C) 15% (D) 45% (E) 90%

Here, $L = 0.9 = 90\%$, so according to this model, only 90% will become infected. This leaves only 10% who will be not infected

D 3. $\displaystyle\int_2^3 \frac{3}{(x-1)(x+2)}\,dx = I$

(A) $-\dfrac{33}{20}$ (B) $-\dfrac{9}{20}$ (C) $\ln\left(\dfrac{5}{2}\right)$ (D) $\ln\left(\dfrac{8}{5}\right)$ (E) $\ln\left(\dfrac{2}{5}\right)$

$$I = \int_2^3 \left[\frac{1}{x-1} - \frac{1}{x+2}\right] dx$$

$$= \Big[\ln|x-1| - \ln|x+2|\Big]_2^3$$

$$(\ln 2 - \ln 5) - (\ln 1 - \ln 4)$$

$$\ln \tfrac{2}{5} + \ln 4$$

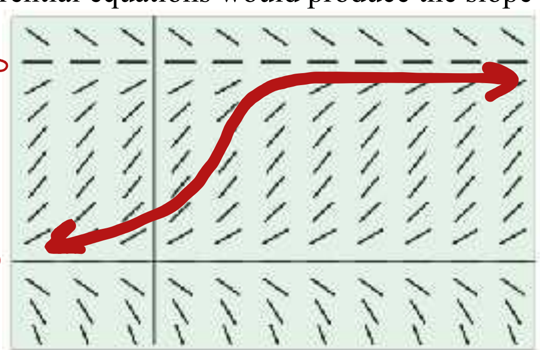
$$\ln\left(\tfrac{2}{5}\cdot 4\right)$$

$$\ln \tfrac{8}{5}$$

B 4. Which of the following differential equations would produce the slope field shown below?

$y=120$

$y=0$



Logistic, so $\dfrac{dy}{dt} = ky(L-y)$

[–3, 8] by [–50, 150]

(A) $\dfrac{dy}{dx} = 0.01x(120-x)$ (B) $\dfrac{dy}{dx} = 0.01y(120-y)$ (C) $\dfrac{dy}{dx} = 0.01y(100-x)$

(D) $\dfrac{dy}{dx} = \dfrac{120}{1+60e^{-1.2x}}$ (E) $\dfrac{dy}{dx} = \dfrac{120}{1+60e^{-1.2y}}$

E 5. The population $P(t)$ of a species satisfies the logistic differential equation $\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$, where the initial population is $P(0) = 3000$ and t is the time in years. What is $\lim_{t\to\infty} P(t)$?

(A) 2500 (B) 3000 (C) 4200 (D) 5000 (E) 10,000

$\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$, need $\frac{dP}{dt} = kP(L-P)$

*factor out a $\frac{1}{5000}$: $\frac{dP}{dt} = \frac{1}{5000}P(10000 - P)$

So, $L = 10,000$

C 6. Suppose a population of wolves grows according to the logistic differential equation $\frac{dP}{dt} = 3P - 0.01P^2$, where P is the number of wolves at time t, in years. Which of the following statements are true?

I. $\lim_{t\to\infty} P(t) = 300$ ✓

II. The growth rate of the wolf population is greatest when $P = 150$. ✓

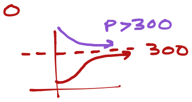
III. If $P > 300$, the population of wolves is increasing. X (decreasing)

(A) I only (B) II only (C) I and II only (D) II and III only (E) I, II, and III

*Factor out $0.01P = \frac{1}{100}P$

$\frac{dP}{dt} = 3(0.01)P(300 - P)$

So, $L = 300$



A

7. $\int \frac{7x}{(2x-3)(x+2)}dx =$

(A) $\frac{3}{2}\ln|2x-3|+2\ln|x+2|+C$ (B) $3\ln|2x-3|+2\ln|x+2|+C$ (C) $3\ln|2x-3|-2\ln|x+2|+C$

(D) $-\frac{6}{(2x-3)^2}-\frac{2}{(x+2)^2}+C$ (E) $-\frac{3}{(2x-3)^2}-\frac{2}{(x+2)^2}+C$

$\int\left[\frac{\frac{7(3/2)}{\frac{3}{2}+2}}{2x-3}+\frac{\frac{-14}{-7}}{x+2}\right]dx$

$\int\left[\frac{\frac{21/2}{7/2}}{2x-3}+\frac{2}{x+2}\right]dx$

$\int\left[\frac{3}{2x-3}+\frac{2}{x+2}\right]dx$

$3(\frac{1}{2})\ln|2x-3|+2\ln|x+2|+c$

D

8. $\int \frac{2x}{x^2+3x+2}dx =$

(A) $\ln|x+2|+\ln|x+1|+C$ (B) $\ln|x+2|+\ln|x+1|-3x+C$ (C) $-4\ln|x+2|+2\ln|x+1|+C$

(D) $4\ln|x+2|-2\ln|x+1|+C$ (E) $2\ln|x|+\frac{2}{3}x+\frac{1}{2}x^2+C$

$\int\frac{2x}{(x+2)(x+1)}dx$

$\int\left[\frac{4}{x+2}-\frac{2}{x+1}\right]dx$

$4\ln|x+2|-2\ln|x+1|+c$

Short Answer/Free Response

Work the following on notebook paper.

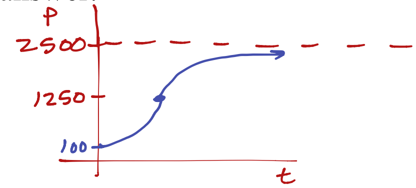
9. Suppose the population of bears in a national park grows according to the logistic differential equation $\frac{dP}{dt} = 5P - 0.002P^2$, where P is the number of bears at time t in years.

   (a) If $P(0) = 100$, then $\lim_{t \to \infty} P(t) =$ 2500. Sketch the graph of $P(t)$. For what values of P is the graph of P increasing? decreasing? Justify your answer.

$\frac{dP}{dt} = 5P - 0.002P^2$ * factor out $0.002P = \frac{2}{1000}P = \frac{1}{500}P$

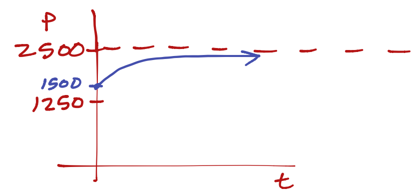
$\frac{dP}{dt} = 0.002P(5(500) - P)$

$\frac{dP}{dt} = 0.002P(2500 - P)$



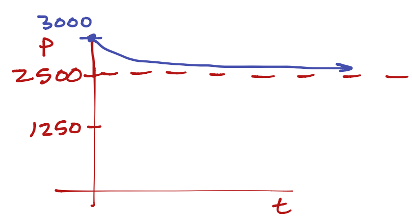
P is increasing for all $t > 0$ & never decreasing since $P(0) = 100 < 2500$

   (b) If $P(0) = 1500$, $\lim_{t \to \infty} P(t) =$ 2500. Sketch the graph of $P(t)$. For what values of P is the graph of P increasing? decreasing? Justify your answer.



P is increasing for all $t > 0$ & never decreasing since $P(0) = 1500 < 2500$

   (c) If $P(0) = 3000$, $\lim_{t \to \infty} P(t) =$ __________. Sketch the graph of $P(t)$. For what values of P is the graph of P increasing? decreasing? Justify your answer.



P is decreasing for all $t > 0$ & never increasing since $P(0) = 3000 > 2500$

   (d) How many bears are in the park when the population of bears is growing the fastest? Justify your answer.

1250 bears, since limit to growth is 2500 bears and 1250 is half of 2500.

10. (Calculator Permitted) A population of animals is modeled by a function P that satisfies the logistic differential equation $\frac{dP}{dt} = 0.01P(100 - P)$, where t is measured in years.

(a) If $P(0) = 20$, solve for P as a function of t.

$P(t) = \frac{L}{1+Ce^{-Lkt}}$ where $L = 100$ & $k = 0.01 = \frac{1}{100}$

So, $P(t) = \frac{100}{1+Ce^{-100\left(\frac{1}{100}\right)t}}$

$P(t) = \frac{100}{1+Ce^{-t}}$

@(0,20): $20 = \frac{100}{1+Ce^{0}}$

$1+C = \frac{100}{20}$

$C = 5-1$

$C = 4$

So, $P(t) = \frac{100}{1+4e^{-t}}$

(b) Use your answer to (a) to find P when $t = 3$ years. Give exact and 3-decimal approximation.

$P(3) = \frac{100}{1+4e^{-3}}$ animals ≈ 83.392 animals

(c) Use your answer to (a) to find t when $P = 80$ animals. Give exact and 3-decimal approximation.

$80 = \frac{100}{1+4e^{-t}}$

$1+4e^{-t} = \frac{100}{80}$

$4e^{-t} = \frac{5}{4} - 1$

$e^{-t} = \frac{1}{4}\left(\frac{1}{4}\right)$

$-t = \ln\left(\frac{1}{16}\right)$

$t = -\ln\left(\frac{1}{16}\right) = \ln(16)$ years

≈ 2.772 years

11. (Calculator Permitted) The rate at which a rumor spreads through a high school of 2000 students can be modeled by the differential equation $\frac{dP}{dt} = 0.003P(2000 - P)$, where P is the number of students who have heard the rumor t hours after 9AM. $K = 0.003$, $L = 2000$

(a) How many students have heard the rumor when it is spreading the fastest?

$\frac{L}{2} = 1000$ students

(b) If $P(0) = 5$, solve for P as a function of t.

$P(t) = \frac{L}{1 + Ce^{-Lkt}}$

$K = 0.003$, $L = 2000$

$Lk = 6$

$P(t) = \frac{2000}{1 + Ce^{-6t}}$

$@(0,5)$: $5 = \frac{2000}{1 + Ce^{0}}$

$1 + C = \frac{2000}{5}$

$C = 400 - 1$

$C = 399$

So, $P(t) = \frac{2000}{1 + 399e^{-6t}}$

(c) Use your answer to (b) to determine how many hours have passed when the rumor is spreading the fastest. Give exact and 3-decimal approximation.

$P(t) = \frac{L}{2}$

$P(t) = 1000$

$1000 = \frac{2000}{1 + 399e^{-6t}}$

$1 + 399e^{-6t} = \frac{2000}{1000}$

$399e^{-6t} = 2 - 1$

$e^{-6t} = \frac{1}{399}$

$-6t = \ln\left(\frac{1}{399}\right)$

$t = -\frac{1}{6}\ln\left(\frac{1}{399}\right)$

$t = \frac{1}{6}\ln(399)$ hours

≈ 0.998 hours

(d) Use your answer to (b) to determine the number of people who have heard the rumor after two hours. Give exact and 3-decimal approximation.

$P(2) = \frac{2000}{1 + 399e^{-6(2)}}$

$P(2) = \frac{2000}{1 + 399e^{-12}}$ students

≈ 1995.108 students

12. Suppose that a population develops according to the logistic equation $\frac{dP}{dt} = 0.05P - 0.0005P^2$ where t is measured in weeks.

* factor out a $0.0005P = \frac{5}{10000}P = \frac{1}{2000}P$

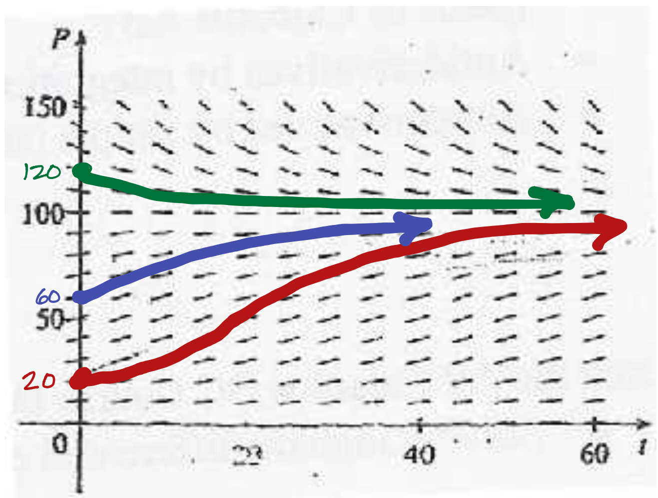
$$\frac{dP}{dt} = 0.0005P(100-P)$$

$K=0.0005,\ L=100$

(a) What is the carrying capacity/limit to growth?

100

(b) A slope field for this equation is shown below.



I. Where are the slopes close to zero?

near $P=0$ and $P=100$

II. Where are they largest?

when $P=50$

III. Which solutions are increasing?

all solutions for $0<P<100$

IV. Which solutions are decreasing?

all solutions for $P>100$

(c) Use the slope field to sketch solutions for initial populations of 20, 60, and 120.

I. What do these solutions have in common?

all these particular solutions approach $P=100$ as $t\to\infty$

II. How do they differ?

the differ in their inc/dec behavior as well as their concavity.

III. Which solutions have inflection points?

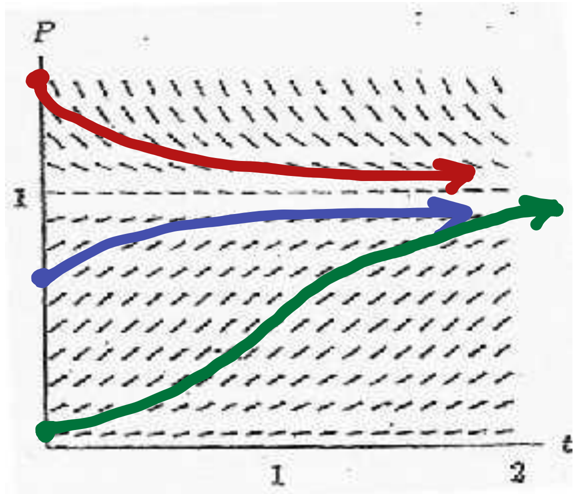
only the solution with $P(0)=20$ shows the inflection value.

IV. At what population level do these inflection points occur?

The inflection point occurs when $P=\frac{L}{2}=50$

13. The slope field show below gives general solutions for the differential equation given by
$\dfrac{dP}{dt} = 3P - 3P^2$.

$\dfrac{dP}{dt} = 3P(1-P)$
$K=3,\ L=1$



(a) On the graph above, sketch three solution curves showing three different types of behavior for the population P.

(b) Describe the meaning of the shape of the solution curves for the population.

I. Where is P increasing?

* If $P(0) \in (0, \frac{1}{2})$, P increases at any increasing rate until $P = \frac{1}{2}$ then increases at a decreasing rate for $P \in (\frac{1}{2}, 1)$.
* If $P(0) \in (\frac{1}{2}, 1)$, P is increasing at a decreasing rate only toward $P = 1$.

II. Where is P decreasing?

* If $P(0) > 1$, then P is decreasing toward $P = 1$

III. What happens in the long run (for large values of t)?

$\lim_{t \to \infty} P(t) = 1$

IV. Are there any inflection points? If so, where?

Inflection points appear on any solution with $P(0) \in (0, \frac{1}{2})$

V. What do the inflection points mean for the population?

This is where the population is growing at the fastest rate.

14. (Calculator Permitted) Newton's Law of Cooling: Newton's Law of Cooling states that the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium. A detective finds a murder victim at 9 A.M. The temperature of the body is measured at $90.3^\circ F$. One hour later, the temperature of the body is $89.0^\circ F$. The temperature of the room has been maintained at a constant $68.0^\circ F$

(a) Assuming the temperature, *T*, of the body obeys Newton's Law of Cooling, write a differential equation for *T*, in degrees Fahrenheit, as a function of *t* hours.

$$\frac{dT}{dt} = kT(L-T)$$, where $L = 68^\circ F$ (Room Temp)

(b) Solve the differential equation to estimate the time the murder occurred.

$$T(t) = \frac{L}{1+Ce^{-Lkt}}$$

Let $t=0$ correspond with 9AM

So, (t, T)

$(0, 90.3)$ & $(1, 89)$

$$T(t) = \frac{68}{1+Ce^{-68kt}}$$

$@(0,90.3)$: $90.3 = \dfrac{68}{1+C}$

$$1+C = \frac{68}{90.3}$$

$$C = \frac{-223}{903}$$

So, $T(t) = \dfrac{68}{1-\frac{223}{903}e^{-68kt}}$

$@(1,89)$: $89 = \dfrac{68}{1-\frac{223}{903}e^{-68k}}$

$$1-\frac{223}{903}e^{-68k} = \frac{68}{89}$$

$$-\frac{223}{903}e^{-68k} = -\frac{21}{89}$$

$$e^{-68k} = \left(-\frac{21}{89}\right)\left(-\frac{903}{223}\right)$$

$$-68k = \ln\left(\frac{18963}{19847}\right)$$

$$k = -\frac{1}{68}\ln\left(\frac{18963}{19847}\right)$$

So, $T(t) = \dfrac{68}{1-\frac{223}{903}e^{(-68)\left(-\frac{1}{68}\ln\left(\frac{18963}{19847}\right)\right)t}}$

$$T(t) = \frac{68}{1-\frac{223}{903}e^{\ln\left(\frac{18963}{19847}\right)t}}$$

Normal Body Temp is $98.6^\circ F$

(c) Call the cops and let them know.

$$98.6 = \frac{68}{1-\frac{223}{903}e^{\ln\left(\frac{18963}{19847}\right)t}}$$

$$1-\frac{223}{903}\cdot\left(\frac{18963}{19847}\right)^t = \frac{68}{98.6}$$

$$\left(\frac{18963}{19847}\right)^t = \frac{8127}{6467}$$

$$t\ln\left(\frac{18963}{19847}\right) = \ln\left(\frac{8127}{6467}\right)$$

$$t = \frac{\ln\left(\frac{8127}{6467}\right)}{\ln\left(\frac{18963}{19847}\right)} \text{ hrs} = -5.014 \text{ hrs} \approx -5 \text{ hrs}$$

So, the murder was around $9-5 = 4$AM