Name KEY Date________________________ Period______

Worksheet 6.5—L'Hôpital's Rule and Indeterminate Forms
Show all work. No calculator at all. Not one keystroke.

Multiple Choice

C 1. Which of the following gives the value of $\lim_{x\to 0} \frac{x}{\tan x}$? $0/0$

(A) -1 (B) 0 (C) 1 (D) π (E) Does not exist

L'Höp it! $\overset{H}{=}$ $\lim_{x\to 0} \frac{1}{\sec^2 x}$

$\frac{1}{1^2}$

1

D 2. Which of the following gives the value of $\lim_{x\to 1} \frac{1-1/x}{1-1/x^2}$? $\frac{0}{0}$

(A) Does not exist (B) 2 (C) 1 (D) 1/2 (E) 0

$\lim_{x\to 1} \frac{1-x^{-1}}{1-x^{-2}}$

L'Höp it! $\overset{H}{=}$ $\lim_{x\to 1} \frac{x^{-2}}{2x^{-3}}$

$\lim_{x\to 1} \frac{x^3}{2x^2}$

$\lim_{x\to 1} \frac{x}{2}$

$\frac{1}{2}$

B 3. Which of the following gives the value of $\lim_{x\to\infty} \frac{\log_2 x}{\log_3 x}$? $\frac{\infty}{\infty}$

(A) 1 (B) $\frac{\ln 3}{\ln 2}$ (C) $\frac{\ln 2}{\ln 3}$ (D) $\ln\left(\frac{3}{2}\right)$ (E) $\ln\left(\frac{2}{3}\right)$

L'Höp it! $\overset{H}{=}$ $\lim_{x\to\infty} \frac{\frac{1}{x\ln 2}}{\frac{1}{x\ln 3}}$

$\lim_{x\to\infty} \frac{\cancel{x}\ln 3}{\cancel{x}\ln 2}$

$\frac{\ln 3}{\ln 2}$

E 4. Which of the following gives the value of $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x}$? 1^∞

(A) 0 (B) 1 (C) e (D) e^2 (E) e^3

Let $y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x}$

$\ln y = \lim_{x\to\infty} 3x\ln\left(1+\frac{1}{x}\right)$ $\infty\cdot 0$

$\ln y = \lim_{x\to\infty} \frac{3\ln\left(1+\frac{1}{x}\right)}{x^{-1}}$ $\frac{0}{0}$

L'Hôp it! $\ln y = \lim_{x\to\infty} \frac{\left(\frac{3}{1+\frac{1}{x}}\right)\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}}$

$\ln y = 3$

$y = e^3$

B 5. What is $\lim_{h\to 0}\frac{8\left(\frac{1}{2}+h\right)^8 - 8\left(\frac{1}{2}\right)^8}{h}$?

(A) 0 (B) 1/2 (C) 1 (D) Does not exist (E) Cannot be determined from the information given

Method 1: $\frac{0}{0}$

L'Hôp it! $\lim_{h\to 0}\frac{64\left(\frac{1}{2}+h\right)^7}{1}$

$64\left(\frac{1}{2}\right)^7$

$\frac{64}{64\cdot 2}$

$\frac{1}{2}$

Method 2:

$\lim_{h\to 0}\frac{8\left(\frac{1}{2}+h\right)^8 - 8\left(\frac{1}{2}\right)^8}{h} = f'\left(\frac{1}{2}\right)$ (Modified Def of $f'(x)$)

for $f(x) = 8x^8$

so, $f'(x) = 64x^7$

$f'\left(\frac{1}{2}\right) = 64\left(\frac{1}{2}\right)^7$

$f'\left(\frac{1}{2}\right) = \frac{1}{2}$

D 6. What is $\lim_{x\to 0}\frac{e^{2x}-1}{\tan x}$? $0/0$

(A) -1 (B) 0 (C) 1 (D) 2 (E) Does not exist

L'Hôp it! $\lim_{x\to 0}\frac{2e^{2x}}{\sec^2 x}$

$\frac{2}{1}$

2

C 7. What is $\lim_{m\to 0} \frac{1}{m}\ln\left(\frac{2+m}{2}\right)$?

(A) e^2 (B) 1 (C) 1/2 (E) 0 (E) Does not exist

Method 1:

$\lim_{m\to 0} \frac{\ln\left(\frac{2+m}{2}\right)}{m} \overset{0/0}{=} \lim_{m\to 0} \frac{\ln(1+\frac{1}{2}m)}{m}$

L'Hôp it!

$\lim_{m\to 0} \frac{\left(\frac{1}{1+\frac{1}{2}m}\right)\left(\frac{1}{2}\right)}{1}$

$\frac{1}{2}$

Method 2

$\lim_{m\to 0} \frac{\ln\left(\frac{2+m}{2}\right)-\ln\left(\frac{2}{2}\right)}{m} = f'(2)$

for $f(x)=\ln\left(\frac{x}{2}\right)$

so, $f'(x) = \frac{1}{x/2}\left(\frac{1}{2}\right) = \frac{1}{x}$

$f'(2)=\frac{1}{2}$ (Modified def)

D 8. What is $\lim_{n\to\infty} \frac{4n^2}{10{,}000n+n^2}$?

(A) 0 (B) $\frac{1}{2{,}500}$ (C) 1 (D) 4 (E) Does not exist

$\lim_{n\to\infty} \frac{4n^2}{1n^2+10000n} \quad \frac{\infty}{\infty}$

$\frac{4}{1}$

4

E 9. $\lim_{x\to 0}(1+2x)^{\csc x} =$ 1^∞

(A) 0 (B) 1 (C) 2 (D) e (E) e^2

Let $y=\lim_{x\to 0}(1+2x)^{\csc x}$

$\ln y = \lim_{x\to 0}(\csc x)\ln(1+2x) \quad \infty\cdot 0$

$\ln y = \lim_{x\to 0}\frac{\ln(1+2x)}{\sin x} \quad \frac{0}{0}$

L'Hôp it!

$\ln y = \lim_{x\to 0}\frac{\left(\frac{1}{1+2x}\right)(2)}{\cos x}$

$\ln y = 2$

$y = e^2$

Free Response

Find the following limits. Show all work and use L'Hopital's Rule whenever possible. Express your answer in exact form. For example, if the value of the limit is π do not write 3.14

10. $\lim\limits_{\theta\to 0}\dfrac{\arctan\theta}{2\theta}$ $0/0$

L'Hôp it! $\lim\limits_{\theta\to 0}\dfrac{\frac{1}{1+\theta^2}}{2}$

$\frac{1}{2}$

11. $\lim\limits_{\theta\to\pi^+}\dfrac{1}{\sin(\theta-\pi)}$ $\frac{1}{0}$

$+\infty$

(or DNE)

12. $\lim\limits_{x\to\pi^+}\dfrac{2x-2\pi}{\sin(x-\pi)}$ $\frac{0}{0}$

$\lim\limits_{x\to\pi^+}\dfrac{2}{\cos(x-\pi)}$

2

13. $\lim\limits_{z\to\pi^+}\sin\left(\dfrac{1}{z-\pi}\right)$

$\sin\left(\lim\limits_{z\to\pi^+}\dfrac{1}{z-\pi}\right)$

$\lim\limits_{z\to\infty}\sin z$

DNE

(oscillation)

14. $\lim\limits_{\alpha\to 0}\alpha\cdot\cot(2\alpha)$ $0\cdot\infty$

$\lim\limits_{\alpha\to 0}\dfrac{\alpha}{\tan(2\alpha)}$ $\frac{0}{0}$

L'Hôp it! $\lim\limits_{\alpha\to 0}\dfrac{1}{2\sec^2(2\alpha)}$

$\frac{1}{2}$

15. $\lim\limits_{y\to\infty}y\cdot\ln\left(\dfrac{y+1}{y-1}\right)$ $\infty\cdot 0$

$\lim\limits_{y\to\infty}\dfrac{\ln\left(\frac{y+1}{y-1}\right)}{y^{-1}}$ $\frac{0}{0}$

L'Hôp it! $\lim\limits_{y\to\infty}\dfrac{\frac{1}{\frac{y+1}{y-1}}\cdot\left(\frac{(y-1)-(y+1)}{(y-1)^2}\right)}{-y^{-2}}$

$\lim\limits_{y\to\infty}\dfrac{\left(\frac{\cancel{y-1}}{y+1}\right)\left(\frac{-2}{(y-1)^{\cancel{2}}}\right)}{-y^{-2}}$

$\lim\limits_{y\to\infty}\dfrac{2y^2}{(y+1)(y-1)}$

$\lim\limits_{y\to\infty}\dfrac{2y^2}{y^2-1}$

2

*careful here with 1-sided limit!!

16. $\lim\limits_{x\to 0^-}x^3\cdot e^{1/x}$

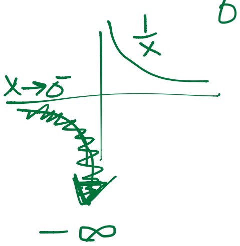
$\lim\limits_{x\to 0^-}x^3\cdot e^{\lim\limits_{x\to 0^-}\left(\frac{1}{x}\right)}$

$0\cdot e^{-\infty}$

$0\cdot\dfrac{1}{e^{\infty}}$

$0\cdot 0$

0



17. $\lim\limits_{t\to\infty}\left(1+\dfrac{3}{t}\right)^t$ 1^∞

$y=\lim\limits_{t\to\infty}\left(1+\frac{3}{t}\right)^t$

$\ln y=\lim\limits_{t\to\infty}t\ln\left(1+\frac{3}{t}\right)$ $\infty\cdot 0$

$\ln y=\lim\limits_{t\to\infty}\dfrac{\ln\left(1+\frac{3}{t}\right)}{t^{-1}}$ $\frac{0}{0}$

L'Hôp it! $\ln y=\lim\limits_{t\to\infty}\dfrac{\left(\frac{1}{1+\frac{3}{t}}\right)(-3t^{-2})}{-t^{-2}}$

$\ln y=\lim\limits_{t\to\infty}\dfrac{3}{1+\frac{3}{t}}$

$\ln y=3$

$y=e^3$

* 18. $\lim\limits_{w\to\infty}\left(\ln w-\sqrt{w}\right)$ $\infty-\infty$

Fun One!

Using $\ln e^x=x$,

Let $\sqrt{w}=\ln e^{\sqrt{w}}$

so, $\lim\limits_{w\to\infty}\left(\ln w-\ln e^{\sqrt{w}}\right)$ $\infty-\infty$

Condense: $\lim\limits_{w\to\infty}\ln\left(\dfrac{w}{e^{\sqrt{w}}}\right)$

$\ln\left(\lim\limits_{w\to\infty}\dfrac{w}{e^{\sqrt{w}}}\right)$

$e^{\sqrt{w}}$ grows faster than w, so limit goes to 0 from pos. side.

$\lim\limits_{w\to 0^+}\ln w$

$-\infty$

(or DNE)

19. $\lim_{t\to\infty} \dfrac{\sin\left(\frac{1}{t}\right)}{\ln t}$ $\dfrac{0}{\infty}$ $= 0\cdot 0$ or $\dfrac{1}{\infty\cdot\infty}$

0

(Not indeterminate)

20. $\lim_{v\to\infty} \dfrac{v^2}{e^{-v}}$ $\dfrac{\infty}{0}$ $= \infty\cdot\infty$ or $\dfrac{1}{0\cdot 0}$

∞

(or DNE)

21. $\lim_{x\to 0^+} \dfrac{x^2 \cdot \sin\left(\frac{1}{x}\right)}{\sin x}$ $\dfrac{0\cdot\boxed{\sin\infty}}{0}$

$\dfrac{0\cdot \text{DNE}}{0}$ by oscillation

DNE

22. $\lim_{u\to 0^+} \left(2^u - 1\right)\sqrt{u}$

$(2^0-1)\sqrt{0}$

$0\cdot 0$

0

(Never Not First!)

23. $\lim_{x\to\infty} \left(\ln|2x-4| - \ln|x+3|\right)$ $\infty - \infty$

$\lim_{x\to\infty} \ln\left|\dfrac{2x-4}{x+3}\right|$

$\ln\left|\lim_{x\to\infty}\dfrac{2x-4}{x+3}\right|$

$\ln|2|$

$\ln 2$

24. $\lim_{\theta\to 0} \pi^2 \dfrac{\tan 2\theta}{\theta\cos 2\theta}$ $0/0$

$\pi^2 \lim_{\theta\to 0} \dfrac{\tan(2\theta)}{\theta\cdot\cos(2\theta)}$

$\dfrac{\text{L'Hôp it!}}{\cancel{H}}$ $\pi^2\cdot \lim_{\theta\to 0} \dfrac{2\sec^2(2\theta)}{\cos(2\theta) - 2\theta\sin(2\theta)}$ product rule

$\pi^2\cdot\dfrac{2}{1-0}$

$2\pi^2$

25. (AP 2010-5) Consider the differential equation $\frac{dy}{dx} = 1 - y$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(1) = 0$. For this particular solution, $f(x) < 1$ for all values of x.

(a) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

$\Delta x = \frac{0-1}{2}$

$\Delta x = -\frac{1}{2}$

x	y	$m = \frac{dy}{dx}$	$\Delta y = \Delta x \cdot m$	$y_{new} = y + \Delta y$
1	0	1	$-1/2$	$-1/2$
$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{4}$	$-\frac{5}{4}$
0	$-\frac{5}{4}$			

✓1

So, $f(0) \approx -\frac{5}{4}$ ✓2

(b) Find $\lim_{x \to 1} \frac{f(x)}{x^3 - 1}$. Show the work that leads to your answer.

$\frac{f(1)}{1-1} = \frac{0}{0}$

L'Hôp it! $\lim_{x \to 1} \frac{f'(x)}{3x^2}$ ✓3

$\frac{f'(1)}{3}$ $\quad f'(1) = \left.\frac{dy}{dx}\right|_{(1,0)} = 1 - 0 = 1$

$\frac{1}{3}$ ✓4

(c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = 1 - y$ with the initial condition $f(1) = 0$.

$\frac{dy}{dx} = 1 - y$

$\frac{1}{1-y}\,dy = dx$ ✓5

correction! $\int \frac{1}{1-y}\,dy = \int 1\,dx$

$-\ln|1-y| = x + C$ ✓4

✓6 $\ln|1-y| = -x + C$

$|1-y| = e^{-x+C}$

$1 - y = Ce^{-x}$

$y = 1 - Ce^{-x}$

@ $(1,0)$: $0 = 1 - Ce^{-1}$ ✓8

$\frac{C}{e} = 1$

$C = e$

So, $y = 1 - e \cdot e^{-x}$ ✓9

or

$y = 1 - e^{1-x}$

26. (2016-BC4)

Consider the differential equation $\frac{dy}{dx} = x^2 - \frac{1}{2}y$.

(a) Find $\frac{d^2y}{dx^2}$ in terms of x and y.

$\frac{d}{dx}$: $\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\frac{dy}{dx}$ ✓1

$\frac{d^2y}{dx^2} = 2x - \frac{1}{2}\left(x^2 - \frac{1}{2}y\right)$ ✓2

(b) Let $y = f(x)$ be the particular solution to the given differential equation whose graph passes through the point $(-2, 8)$. Does the graph of f have a relative minimum, a relative maximum, or neither at the point $(-2, 8)$? Justify you answer.

$\frac{dy}{dx} = x^2 - \frac{1}{2}y$ ✓3

$\left.\frac{dy}{dx}\right|_{(-2,8)} = (-2)^2 - \frac{1}{2}(8) = 4 - 4 = 0$

So, $(-2,8)$ is a critical value of $y=f(x)$.

$\left.\frac{d^2y}{dx^2}\right|_{(-2,8)} = 2(-2) - \frac{1}{2}(0) = -4 < 0$ $\frown$

So, $y=f(x)$ has a relative max @ $(-2,8)$ ✓4

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$.

Find $\lim_{x\to -1}\left(\frac{g(x)-2}{3(x+1)^2}\right)$. Show the work that leads to your answer.

$= \frac{g(-1)-2}{3(-1+1)^2}$ $\frac{0}{0}$

L'Hôp it ✓5

$\lim_{x\to -1} \frac{g'(x)}{6(x+1)\cdot 1}$ $\frac{g'(-1)}{0}$ $\frac{0}{0}$

$g'(-1) = \left.\frac{dy}{dx}\right|_{(-1,2)} = 1 - 1 = 0$

L'Hôp it again! $g''(-1) = \left.\frac{d^2y}{dx^2}\right|_{(-1,2)} = -2$

$\lim_{x\to -1} \frac{g''(-1)}{6}$ ✓6

$-\frac{2}{6} = -\frac{1}{3}$ ✓7

(d) Let $y = h(x)$ be the particular solution to the given differential equation with $h(0) = 2$.

(BC) Use Euler's method starting at $x = 0$ with two step of equal size, to approximate $h(1)$.

(AB) Use the linearization of $h(x)$ at $x = 0$ to approximate $h(1)$.

(BC) $\Delta x = \frac{1}{2}$

x	y	$\frac{dy}{dx} = m$	$\Delta y = m\Delta x$	$y_{new} = y + \Delta y$
0	2	-1	$-1/2$	$3/2$
$1/2$	$3/2$	$\frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$	$-1/4$	$5/4$ ✓8
1	$5/4$			

$\Delta x = \frac{b-a}{n} = \frac{1-0}{2} = \frac{1}{2}$

$\frac{dy}{dx} = x^2 - \frac{1}{2}y$ So, $h(1) \approx \frac{5}{4}$ ✓9

(AB Version) $h(0) = 2$, pt: $(0,2)$

$h'(0) = \left.\frac{dy}{dx}\right|_{(0,2)} = 0 - 1 = -1 = m$

So, $\mathcal{L}(x) = 2 - 1(x - 0)$

$\mathcal{L}(x) = 2 - x$

$h(1) \approx \mathcal{L}(1) = 2 - 1 = 1$