Name KEY Date Period

Worksheet 6.6—Improper Integrals
Show all work. No calculator unless explicitly stated.

Short Answer
1. Classify each of the integrals as proper or improper integrals. Give a clear reason for each.

(a) $\int_5^\infty \frac{dx}{(x-2)^2}$

Improper. ∞ as upper limit of integration

(b) $\int_1^5 \frac{dx}{(x-2)^2}$

Improper. VA @ $x=2 \in [1,5]$, the interval of integration.

(c) $\int_2^5 \frac{dx}{(x-2)^2}$

Improper: VA @ $x=2$, the lower limit of integration

(d) $\int_3^5 \frac{dx}{(x-2)^2}$

Proper. VA @ $x=2 \notin [3,5]$, outside the interval of integration.

2. Answer the following.

(a) If $\int_a^\infty f(x)dx = K$ and $0 < g(x) \le f(x)$, what can we say about $\int_a^\infty g(x)dx$?

Since integral of $f(x)$ converges, by comparison, anything smaller than it, over the same interval, will converge too. Since $g(x) \le f(x)$, $\int_a^\infty g(x)dx$ converges (but not necessarily to K).

(b) If $\int_a^\infty f(x)dx = K$ and $0 < f(x) < g(x)$, what can we say about $\int_a^\infty g(x)dx$?

If the integral of $f(x)$ converges to K, we cannot say anything for sure about an integral over the same interval of a larger function. Since $g(x) > f(x)$, $\int_a^\infty g(x)dx$ may either converge OR diverge.

(c) If $\int_a^\infty f(x)dx$ diverges and $0 < f(x) \le g(x)$, what can we say about $\int_a^\infty g(x)dx$?

If the integral of $f(x)$ diverges, then any integral over the same interval of a larger function will diverge as well.

(d) If $\int_a^\infty f(x)dx$ diverges and $0 < g(x) < f(x)$, what can we say about $\int_a^\infty g(x)dx$?

if the integral of the larger function diverges, then the integral of the smaller function on the same interval may diverge OR converge, so we cannot say anything conclusively.

3. If $\int_1^{\infty} \frac{1}{x^p}\,dx$ converges for $p>1$, what can be said in general about improper integrals of the form $\int_a^{\infty} \frac{1}{x^p}\,dx$? For what values of *a* does the function diverge? Converge? To what ?

$\int_1^\infty \frac{1}{x^p}dx$, for $p>1$, converges to $\frac{1}{p-1}$. For $\int_a^\infty \frac{1}{x^p}$, $p>1$, still converges for $p>1$, $a>0$, but not necessarily to $\frac{1}{p-1}$. We still call these integrals ($\int_a^\infty \frac{1}{x^p}dx, a>0$) p-series integrals.

if $p \le 1$, the integral diverges, as the graph of $y=\frac{1}{x^p}$ does NOT go toward the x-axis (horizontal asymptote) fast enough (steep enough) to cause the integral to converge to a finite value.

4. Determine if the improper integral converges or diverges by finding a function to compare it to. Justify by showing the inequality and discussing the convergence/divergence of the function to which you compare.

(a) $\int_2^{\infty} \frac{x^5}{x^6-1}\,dx$

* $\lim_{x\to\infty} \frac{x^5}{x^6-1} = 0$ (HA @ y=0)

$\frac{x^5}{x^6-1} \sim \frac{1}{x} \to$ Diverges

So we need a smaller function

$\frac{x^5}{x^6} \le \frac{x^5}{x^6-1}$, $\forall x \ge 2$

So, $\int_2^\infty \frac{x^5}{x^6}dx = \int_2^\infty \frac{1}{x}dx$ which is a divergent p-series with $p=1\le 1$.

So, $\int_2^\infty \frac{x^5}{x^6-1}dx$ diverges too!

(b) $\int_2^{\infty} \frac{x^3+1}{(x^4+4x+1)^2}\,dx$

* $\lim_{x\to\infty} \frac{x^3+1}{(x^4+4x+1)^2} = 0$ (HA @ y=0)

$\frac{x^3}{(x^4+4x+1)^2} \sim \frac{x^3}{x^8} \sim \frac{1}{x^5}$

which converges, so we need a bigger function that $y=\frac{1}{x^5}$

$\frac{x^3+1}{x^8+\dots} \le \frac{1}{x^5}$, $\forall x \ge 2$

$\int_2^\infty \frac{1}{x^5}dx$ is a convergent p-series with $p=5>1$,

So, $\int_2^\infty \frac{x^3+1}{(x^4+4x+1)^2}dx$ converges as well!

(c) $\int_1^{\infty} \frac{dx}{(x+5)^5}$

* $\lim_{x\to\infty} \frac{1}{(x+5)^5} = 0$ (HA @ y=0)

$\frac{1}{(x+5)^5} \sim \frac{1}{x^5}$ Which Converges.

So, We need a bigger function.

$\frac{1}{(x+5)^5} \le \frac{1}{x^5}$, $\forall x \ge 1$

So, $\int_1^\infty \frac{1}{x^5}dx$ is a convergent p-series with $p=5>1$.

So, $\int_1^\infty \frac{1}{(x+5)^5}dx$ Converges also!

(d) $\int_4^{\infty} \frac{3+\sin x}{x}\,dx$

* $\lim_{x\to\infty} \frac{3+\sin x}{x} = 0$ (HA @ y=0)

$\frac{3+\sin x}{x} \sim \frac{1}{x}$ which Diverges

So we need a smaller function

$\frac{1}{x} \le \frac{3+\sin x}{x}$, $\forall x \ge 4$

(Range of $3+\sin x$ is $[2,4] > 1$)

So, $\int_4^\infty \frac{1}{x}dx$ is a divergent p-series with $p=1\le 1$.

So, $\int_4^\infty \frac{3+\sin x}{x}dx$ diverges to boot!

Multiple Choice

C 5. $\int_0^{\infty} x^2 e^{-x^3}\,dx =$

(A) $-\frac{1}{3}$ (B) 0 (C) $\frac{1}{3}$ (D) 1 (E) Diverges

$\lim_{b\to\infty} \int_0^b x^2 e^{(-x^3)}dx$ $\quad -3x^2$

$\lim_{b\to\infty} \left(-\frac{1}{3}\right) e^{-x^3}\Big|_0^b$

$\lim_{b\to\infty} -\frac{1}{3}\left[e^{-b^3} - 1\right]$

$-\frac{1}{3}\left[0-1\right]$

$\frac{1}{3}$

FANTASTIC AP EXAM QUESTION!

C 6. Which of the following gives the value of the integral $\int_1^{\infty} \frac{dx}{x^{1.01}}$?

(A) 1 (B) 10 (C) 100 (D) 1000 (E) Diverges

Convergent p-series ($p=1.01>1$) Starting at $x=1$,

So, $\int_1^{\infty} \frac{1}{x^{1.01}}\,dx = \frac{1}{1.01-1}$

$= \frac{1}{\frac{1}{100}}$

$= 100$

B 7. Which of the following gives the value of the integral $\int_0^1 \frac{dx}{x^{0.5}}$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) Diverges

$\int_0^1 \frac{1}{x^{1/2}}\,dx$ for $f(x)=\frac{1}{\sqrt{x}}$, $x\neq 0$, f has a VA @ $x=0$

$\lim_{b\to 0^+} \int_b^1 x^{-1/2}\,dx$

$\lim_{b\to 0^+} 2x^{1/2}\Big|_b^1$

$\lim_{b\to 0^+} 2[1-\sqrt{b}]$

$2[1-0]$

2

E 8. Which of the following gives the value of the integral $\int_0^1 \frac{dx}{x-1}$?

(A) -1 (B) $-1/2$ (C) 0 (D) 1 (E) Diverges

Method 1:

$\int_0^1 \frac{1}{x-1}\,dx$, for $f(x)=\frac{1}{x-1}$, $x\neq 1$ & f has a VA @ $x=1$

$\lim_{b\to 1^-} \int_0^b \frac{1}{x-1}\,dx$

$\lim_{b\to 1^-} \ln|x-1|\Big|_0^b$

$\lim_{b\to 1^-} [\ln|b-1| - \ln|-1|]$

$\lim_{b\to 1^-} [\ln|b-1| - 0]$

$-\infty$

Diverges

Method 2:

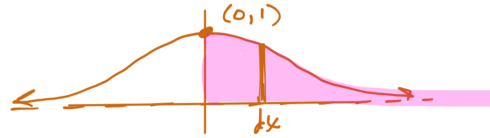
Intuition

$\frac{1}{x-1} \sim \frac{1}{x}$

which Diverges as we approach BOTH the Horz. Asympt. & the Vert. Asympt., so DIVERGES

C 9. Which of the following gives the value of the area under the curve $y = \dfrac{1}{x^2+1}$ in the first quadrant?

(A) $\dfrac{\pi}{4}$ (B) 1 (C) $\dfrac{\pi}{2}$ (D) π (E) Diverges



X-TRA SPACE!

$$\text{Area} = \int_0^\infty \left(\frac{1}{x^2+1} - 0\right) dx$$

$$\lim_{b\to\infty} \arctan x \Big|_0^b$$

$$\lim_{b\to\infty} [\arctan b - \arctan 0]$$

$$\frac{\pi}{2} - 0$$

$$\frac{\pi}{2}$$

C 10. Determine if $\int_0^2 f(x)dx$ is convergent or divergent when $f(x) = \begin{cases} x^{-1/2}, & x \le 1 \\ x, & 1 < x \le 2 \end{cases}$, and if it is convergent, find its value.

(A) 1/2 (B) 5/2 (C) 7/2 (D) 4 (E) Diverges

for $x \le 1$, $f(x) = \frac{1}{\sqrt{x}}$ so $x > 0$ & $x \le 1$

$f(x)$ has a VA @ $x = 0$

$$\int_0^2 f(x)\,dx$$

$$\int_0^1 x^{-1/2}\,dx + \int_1^2 x\,dx$$

$$\lim_{b\to 0^+} \int_b^1 x^{-1/2}\,dx + \int_1^2 x\,dx$$

$$\lim_{b\to 0^+} 2x^{1/2}\Big|_b^1 + \frac{1}{2}x^2\Big|_1^2$$

$$\lim_{b\to 0^+} 2[1-\sqrt{b}] + \frac{1}{2}[4-1]$$

$$2[1-0] + \frac{1}{2}[3]$$

$$2 + \frac{3}{2}$$

$$\frac{7}{2}$$

E 11. $\displaystyle\int_2^\infty \frac{x}{\sqrt[3]{x^2-2}}\,dx =$

(A) $\dfrac{3\cdot 2^{2/3}}{4}$ (B) $2^{2/3}$ (C) $-\dfrac{3\cdot 2^{2/3}}{4}$ (D) $-\dfrac{3\cdot 2^{2/3}}{2}$ (E) Diverges

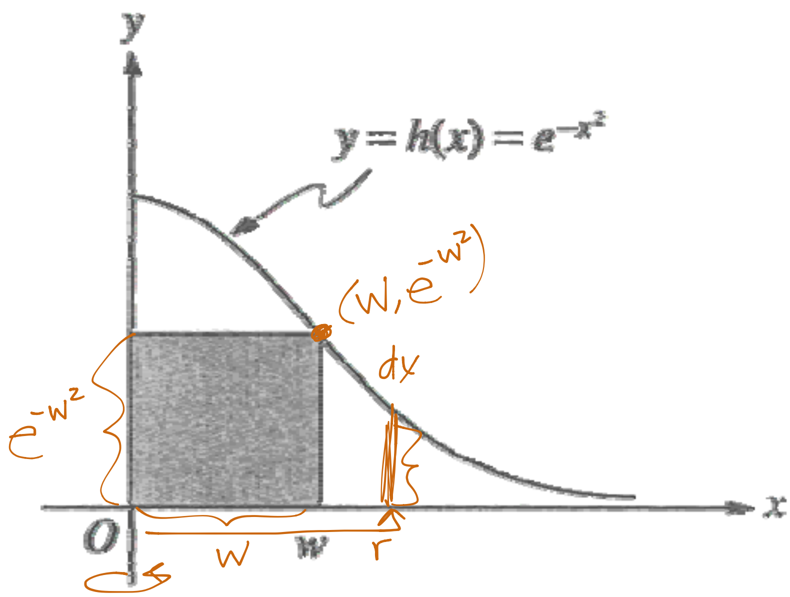
$$\frac{x}{\sqrt[3]{x^2-2}} \sim \frac{x}{x^{2/3}} \sim x^{1/3} = \frac{1}{x^{-1/3}}$$

so $\int_2^\infty \frac{1}{x^{-1/3}}\,dx$ is a divergent p-series with $p = -\frac{1}{3} \le 1$

Free Response

Hmmm... [strokes chin]
OK, done! I accept it!

12. (AP 1996-1) Consider the graph of the function h given by $h(x)=e^{-x^2}$ for $0 \le x < \infty$.



(a) Let R be the unbounded region in the first quadrant below the graph of h. Find the volume of the solid generated when R is revolved about the y-axis.

ParaSHELL

$$V = 2\pi \int_0^{\infty} (\overset{r(x)}{x})(\overset{h(x)}{e^{-x^2}} - 0)\,dx$$

$$V = 2\pi \int_0^{\infty} x \cdot e^{-x^2}\,dx$$

$$V = \lim_{b\to\infty} (2\pi)\left(-\tfrac{1}{2}\right) e^{-x^2}\Big|_0^b$$

$$V = \lim_{b\to\infty} -\pi\left[e^{-b^2} - 1\right]$$

$$V = -\pi[0-1]$$

$$V = \pi$$

(b) Let $A(w)$ be the area of the shaded rectangle shown in the figure. Show that $A(w)$ has its maximum value when w is the x-coordinate of the point of inflection of the graph of h.

Area of rectangle $= A(w) = w \cdot e^{-w^2}$

$A'(w) = 1 \cdot e^{-w^2} + w(-2we^{-w^2}) = 0$

$\boxed{A'(w) = e^{-w^2}(1-2w^2)} = 0$

$e^{-w^2} = 0$ No Solution or $1-2w^2=0$

$2w^2 = 1$

$w^2 = \frac{1}{2}$

$w = -\sqrt{\frac{1}{2}}$ or $w = \sqrt{\frac{1}{2}}$

Not in Quad I

So, $w = \sqrt{\frac{1}{2}}$ is a critical value of $A(w)$

w	0	$\sqrt{1/4}$	$\sqrt{\frac{1}{2}}$	1	$\to \infty$
A'		+		−	

$*$ Note: $\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

since $A'(w) > 0$ $\forall w \in [0, \sqrt{\frac{1}{2}})$ & $A'(w) < 0$ $\forall w > \sqrt{\frac{1}{2}}$, $w = \sqrt{\frac{1}{2}}$ maximizes $A(w)$ absolutely on $w \in [0, \infty)$

$h(x) = e^{-x^2}$

$h'(x) = -2xe^{-x^2}$

$h''(x) = -2e^{-x^2} - 2x(-2xe^{-x^2})$

$h''(x) = -2e^{-x^2} + 4x^2e^{-x^2} = 0$

$-2e^{-x^2}(1-2x^2) = 0$

$e^{-x^2} = 0$ No soln $\quad 1-2x^2 = 0$

$x = \sqrt{\frac{1}{2}}$

x		$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{1}{2}}$	1
h''		−		+

So, h has an inflection pt. at $x = \sqrt{\frac{1}{2}}$ which is the value that maximizes $A(w)$.

13. (AP 2001-5) Let f be the function satisfying $f'(x) = -3xf(x)$, for all real numbers x, with $f(1) = 4$ and $\lim_{x\to\infty} f(x) = 0$.

(a) Evaluate $\int_1^{\infty} -3xf(x)\,dx$. Show the work that leads to your answer.

$\lim_{x\to\infty} \int_1^b -3xf(x)dx$

$\lim_{x\to\infty} \int_1^b f'(x)\,dx$

$\lim_{x\to\infty} f(x)\Big|_1^b$ ✓1

$\lim_{x\to\infty} (f(b) - f(1))$

$0 - 4$

-4 ✓2

(b) Use Euler's method, starting at $x = 1$ with a step size of 0.5, to approximate $f(2)$.

$\Delta x = 0.5 = \frac{1}{2}$

✓3

x	y	$m = \frac{dy}{dx}$	$\Delta y = m\Delta x$	y_{new}
1	4	−12	−6	−2
$1.5 = \frac{3}{2}$	−2	9	4.5	2.5
2	2.5			

$f'(x) = -3x\,f(x)$

$f'(1) = -3(1)\cdot f(1)$
$= -3(4)$
$= -12$

$f'(1.5) = -3(1.5)\,f(1.5)$
$= -4.5(-2)$ ← from table
$= 9$

So, $f(2) \approx 2.5$ ✓4

(c) Write an expression for $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = -3xy$ with the intial condition $f(1) = 4$.

$\frac{dy}{dx} = -3xy$

$\frac{1}{y}dy = -3x\,dx$ ✓5

$\int \frac{1}{y}dy = -3\int x\,dx$

✓6 $\ln|y| = -\frac{3}{2}x^2 + C$ ✓7

$|y| = e^{-\frac{3}{2}x^2 + C}$

$y = Ce^{-\frac{3}{2}x^2}$

for $f(1)=4$: $4 = Ce^{-\frac{3}{2}}$ ✓8

$C = \frac{4}{e^{-3/2}}$

$C = 4e^{3/2}$

So, $y = 4e^{3/2}\cdot e^{-\frac{3}{2}x^2}$ ✓9

14. (AP 2010B-5) Let f and g be the functions defined by $f(x)=\dfrac{1}{x}$ and $g(x)=\dfrac{4x}{1+4x^2}$, for all $x>0$.

(a) Find the absolute maximum value of g on the open interval $(0,\infty)$ if the maximum exists. Find the absolute minimum value of g on the open interval $(0,\infty)$ if the minimum exists. Justify your answers.

y-value

y-value

$$g(x)=\frac{4x}{1+4x^2} \quad \checkmark 1$$

$$g'(x)=\frac{(1+4x^2)(4)-(4x)(8x)}{(1+4x^2)^2} \quad \checkmark 2$$

$$g'(x)=\frac{4(1+4x^2-8x^2)}{(1+4x^2)^2}$$

$$g'(x)=\frac{4(1-4x^2)}{(1+4x^2)^2}$$

$g'(x)=$ DNE Never

$g'(x)=0$ when $4(1-4x^2)=0$

$x^2=\frac{1}{4}$

$x=\sqrt{\frac{1}{4}}$

$x=\frac{1}{2}$ $\checkmark 3$

x	0	$\frac{1}{4}$	$\frac{1}{2}$	1	$\to\infty$
g'		$+$		$-$	

Since $g'>0$ $\forall x\in(0,\frac{1}{2})$ and $g'<0$ $\forall x>\frac{1}{2}$, g has an abs. max at $x=\frac{1}{2}$. This max value is $g(\frac{1}{2})=\dfrac{4(\frac{1}{2})}{1+4(\frac{1}{2})^2}=\dfrac{2}{1+1}=1$ $\checkmark 4$ $\checkmark 5$

g has no minimum values on $x\in(0,\infty)$

(b) Find the area of the unbounded region in the first quadrant to the right of the vertical line $x=1$, below the graph of f, and above the graph of g.

TOP BOTTOM

$$\text{Area}=\int_1^\infty (f(x)-g(x))\,dx$$

$$=\lim_{b\to\infty}\int_1^b (f(x)-g(x))\,dx$$

$$=\lim_{b\to\infty}\int_1^b\left(\frac{1}{x}-\frac{4x}{1+4x^2}\right)dx \quad \checkmark 6$$

$$=\lim_{b\to\infty}\left[\ln|x|-(4)\left(\tfrac{1}{8}\right)\ln|1+4x^2|\right]\Big|_1^b \quad \checkmark 7 \quad \checkmark 8$$

$$=\lim_{b\to\infty}\left[\left(\ln b-\tfrac{1}{2}\ln(1+4b^2)\right)-\left(\ln 1-\tfrac{1}{2}\ln 5\right)\right]$$

$\infty - \infty$

$$=\lim_{b\to\infty}\left[\ln\left(\frac{b}{\sqrt{1+4b^2}}\right)+\tfrac{1}{2}\ln 5\right]$$

*Nice trick: for $b>0$, $b=\sqrt{b^2}$

$$=\lim_{b\to\infty}\left[\ln\left(\frac{\sqrt{b^2}}{\sqrt{1+4b^2}}\right)+\tfrac{1}{2}\ln 5\right]$$

$$=\lim_{b\to\infty}\left[\ln\left(\frac{b^2}{1+4b^2}\right)^{1/2}+\tfrac{1}{2}\ln 5\right]$$

$$=\lim_{b\to\infty}\left[\tfrac{1}{2}\ln\left(\frac{\frac{1}{b^2}b^2}{\frac{1}{b^2}(4b^2+1)}\right)+\tfrac{1}{2}\ln 5\right]$$

$$=\tfrac{1}{2}\ln\left(\tfrac{1}{4}\right)+\tfrac{1}{2}\ln 5 \quad \checkmark 9$$

or

$$\tfrac{1}{2}\left[\ln\left(\tfrac{1}{4}\right)+\ln 5\right]$$

$$\tfrac{1}{2}\ln\left(\tfrac{5}{4}\right)$$