Name KEY Date________________________ Period______

Worksheet 7.1—Intro to Parametric & Vector Calculus
Show all work. No calculator unless explicitly stated.

Short Answer

1. If $x = t^2 - 1$ and $y = e^{t^3}$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$$
$$= \frac{3t^2 e^{t^3}}{2t}$$

2. If a particle moves in the xy-plane so that at any time $t > 0$, its position vector is $\vec{s} = \left\langle \ln\left(t^2 + 5t\right), 3t^2 \right\rangle$, find its velocity vector at time $t = 2$.

$$\vec{v} = \vec{s}\,' = \left\langle \frac{2t+5}{t^2+5t}, 6t \right\rangle$$
$$\vec{v}(2) = \left\langle \frac{4+5}{4+10}, 12 \right\rangle$$
$$= \left\langle \frac{9}{14}, 12 \right\rangle$$

3. A particle moves in the xy-plane so that at any time t, its coordinates are given by $x = t^5 - 1$, $y = 3t^4 - 2t^3$. Find its acceleration vector at $t = 1$.

$$\vec{s}(t) = \langle t^5 - 1, 3t^4 - 2t^3 \rangle$$
$$\vec{s}\,'(t) = \vec{v}(t) = \langle 5t^4, 12t^3 - 6t^2 \rangle$$
$$\vec{s}\,''(t) = \vec{v}\,'(t) = \vec{a}(t) = \langle 20t^3, 36t^2 - 12t \rangle$$
$$\vec{a}(1) = \langle 20, 36 - 12 \rangle$$
$$= \langle 20, 24 \rangle$$

4. If a particle moves in the xy-plane so that at time t, its position vector is $\left\langle \sin\left(3t-\frac{\pi}{2}\right), 3t^2 \right\rangle$, find the velocity vector at time $t=\frac{\pi}{2}$.

$\vec{s} = $

$\vec{s}\,' = \vec{v} = \langle 3\cos(3t-\frac{\pi}{2}), 6t \rangle$

$\vec{v}(\frac{\pi}{2}) = \langle 3\cos(\pi), 3\pi \rangle$

$= \langle -3, 3\pi \rangle$

5. A particle moves on the curve $y=\ln x$ so that its x-component has velocity $x'(t)=t+1$ for $t \geq 0$. At time $t=0$, the particle is at the point $(1,0)$. Find the position of the particle at time $t=1$.

$X(1) = x(0) + \int_0^1 x'(t)\,dt$

$= 1 + \int_0^1 (t+1)\,dt$

$= 1 + \left[\frac{1}{2}t^2 + t\right]\Big|_0^1$

$= 1 + \left[(\frac{1}{2}+1) - (0)\right]$

$X(1) = \frac{5}{2}$

So, when $x = \frac{5}{2}$,

$y(\frac{5}{2}) = \ln\frac{5}{2}$,

So, at $t=1$, the particle is at $(x,y) = (\frac{5}{2}, \ln\frac{5}{2})$.

6. A particle moves in the xy-plane in such a way that its velocity vector is $\langle 1+t, t^3 \rangle$. If the position vector at $t=0$ is $\langle 5,0 \rangle$, find the position of the particle at $t=2$.

$x'(t) = 1+t$

$x(2) = x(0) + \int_0^2 x'(t)\,dt$

$= 5 + \int_0^2 (1+t)\,dt$

$= 5 + \left[t + \frac{1}{2}t^2\right]\Big|_0^2$

$= 5 + \left[(2+2) - (0)\right]$

$= 5+4$

$= 9$

$y'(t) = t^3$

$y(2) = y(0) + \int_0^2 y'(t)\,dt$

$= 0 + \int_0^2 (t^3)\,dt$

$= \frac{1}{4}t^4\Big|_0^2$

$= \frac{1}{4}[16-0]$

$= 4$

So, at $t=2$, the particle is at $(x,y) = (9,4)$

7. A particle moves along the curve $xy = 10$. If $x = 2$ and $\frac{dy}{dt} = 3$, what is the value of $\frac{dx}{dt}$?

$y = \frac{10}{x}$

$\frac{dy}{dx} = \frac{-10}{x^2} = \frac{dy/dt}{dx/dt}$

when $x=2$: $\frac{-10}{(2^2)} = \frac{3}{dx/dt}$

$\frac{\frac{dx}{dt}}{3} = \frac{4}{-10}$

$\frac{dx}{dt} = \left(-\frac{2}{5}\right)(3)$

$\frac{dx}{dt} = -\frac{6}{5}$

8. The position of a particle moving in the xy-plane is given by the parametric equations $x = t^3 - \frac{3}{2}t^2 - 18t + 5$ and $y = t^3 - 6t^2 + 9t + 4$. For what value(s) of t is the particle at rest?

$x'(t) = 3t^2 - 3t - 18 = 0$

$3(t^2 - t - 6) = 0$

$3(t-3)(t+2) = 0$

$t = 3$, $t = -2$

$y'(t) = 3t^2 - 12t + 9 = 0$

$3(t^2 - 4t + 3) = 0$

$3(t-3)(t-1) = 0$

$t = 3$, $t = 1$

BOTH $x'(t)$ & $y'(t) = 0$

So, particle is at rest for $t = 3$.

9. A curve C is defined by the parametric equations $x = t^3$ and $y = t^2 - 5t + 2$. Write an equation of the line tangent to the graph of C at the point $(8, -4)$. $x'(t) = 3t^2$ $y'(t) = 2t - 5$

Find t: when $x = 8$

$t^3 = 8$

$t = 2$

So $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2t-5}{3t^2}$

$\left.\frac{dy}{dx}\right|_{t=2} = \frac{-1}{12}$

So, $\mathcal{L} = -4 - \frac{1}{12}(x - 8)$

10. (Calculator Permitted) A particle moves in the xy – plane so that the position of the particle is given by $x(t) = 5t + 3\sin t$ and $y(t) = (8-t)(1-\cos t)$. Find the velocity vector at the time when the particle's horizontal position is $x = 25$.

$X(t) = 25$
$5t + 3\sin t = 25$
$5t + 3\sin t - 25 = 0$
$t = 5.445\ldots = A$ (store as A)

so, $\vec{V}(A) = \langle x'(A), y'(A) \rangle$
$= \langle 7.008, -2.228 \rangle$

* Remember, numeric derivatives are $\boxed{\text{MATH}}\ \boxed{\#8}$ on calculator.

Free Response: (still)

11. The position of a particle at any time $\boxed{t \geq 0}$ is given by $x(t) = t^2 - 3$ and $y(t) = \frac{2}{3}t^3$.

(a) Find the magnitude of the velocity vector at time $t = 5$.

$x'(t) = 2t$, $y'(t) = 2t^2$
$\vec{V} = \langle 2t, 2t^2 \rangle$
$\vec{V}(5) = \langle 10, 50 \rangle$
$\|\vec{V}(5)\| = \sqrt{10^2 + 50^2}$
$= \sqrt{2600}$
$= 10\sqrt{26}$

(b) Find the $\underbrace{\text{total distance traveled}}_{\text{arc length}}$ by the particle from $t = 0$ to $t = 5$.

$\mathcal{L} = \text{Dist} = \int_0^5 \sqrt{(2t)^2 + (2t^2)^2}\, dt$
$= \int_0^5 \sqrt{4t^2 + 4t^4}\, dt$
$= \int_0^5 \sqrt{4t^2(1+t^2)}\, dt$
$= 2\int_0^5 t(1+t^2)^{1/2}\, dt$
$= 2\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)(1+t^2)^{3/2}\Big|_0^5$
$= \frac{2}{3}\left[26^{3/2} - 1^{3/2}\right]$

(c) Find $\frac{dy}{dx}$ as a function of x. (not t!)

Eliminate the parameter 1st.
$x = t^2 - 3$, $y = \frac{2}{3}t^3$
$x + 3 = t^2$
$\boxed{t \geq 0}: t = +\sqrt{x+3}$
so $y = \frac{2}{3}\left[(x+3)^{1/2}\right]^3$
$y = \frac{2}{3}(x+3)^{3/2}$
$\frac{dy}{dx} = (x+3)^{1/2}$
$= \sqrt{x+3}$

12. Point $P(x, y)$ moves in the xy–plane in such away that $\frac{dx}{dt} = \frac{1}{t+1}$ and $\frac{dy}{dt} = 2t$ for $t \ge 0$.

(a) Find the coordinates of P in terms of t when $t = 1$, $x = \ln 2$, and $y = 0$.

$x(t) = \int \frac{1}{t+1}\,dt$
$= \ln|t+1| + C$

when $t=1, x=\ln 2$: $\ln 2 = \ln 2 + C$
$C = 0$
So, $x(t) = \ln|t+1|$

$y(t) = \int 2t\,dt$
$= t^2 + C$
when $t=1$: $y=0$ $\quad 0 = 1^2 + C$
$C = -1$
So, $y(t) = t^2 - 1$

So $P = (x, y) = (\ln|t+1|, t^2 - 1)$

(b) Write an equation expressing y in terms of x.

$\boxed{t \ge 0}$ eliminate the parameter

$x = \ln|t+1|, \quad y = t^2 - 1$
$e^x = |t+1|$
$e^x = t+1, \; t \ge 0$
$t = e^x - 1$

so $y = (\quad)^2 - 1$
$y = (e^x - 1)^2 - 1$

(c) Find the average rate of change of y with respect to x as t varies from 0 to 4.

$\text{Avg RoC} = \frac{y(4) - y(0)}{4 - 0}$
$= \frac{(4^2 - 1) - (0^2 - 1)}{\ln(4+1) - \ln(0+1)}$
$= \frac{16}{\ln 5}$

(d) Find the instantaneous rate of change of y with respect to x when $t = 1$.

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$
$= \frac{2t}{\frac{1}{t+1}}$
$= 2t(t+1)$

$\left.\frac{dy}{dx}\right|_{t=1} = 2(2)$
$= 4$

13. Consider the curve C given by the parametric equations $x = 2 - 3\cos t$ and $y = 3 + 2\sin t$, for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$.

$x'(t) = 3\sin t \quad y'(t) = 2\cos t$

(a) Find $\frac{dy}{dx}$ as a function of t.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{2\cos t}{3\sin t} = \frac{2}{3}\cot t$$

(b) Find an equation of the tangent line at the point where $t = \frac{\pi}{4}$.

pt: $\left(x\left(\frac{\pi}{4}\right), y\left(\frac{\pi}{4}\right)\right) = \left(2 - 3\cos\frac{\pi}{4}, 3 + 2\sin\frac{\pi}{4}\right) = \left(2 - \frac{3\sqrt{2}}{2}, 3 + \sqrt{2}\right)$

m: $\left.\frac{dy}{dx}\right|_{\frac{\pi}{4}} = \frac{2}{3}\cot\frac{\pi}{4} = \frac{2}{3}$

So, $\mathcal{L}(x) = \left(3 + \sqrt{2}\right) + \frac{2}{3}\left(x - \left(2 - \frac{3\sqrt{2}}{2}\right)\right)$

(c) (Calculator Permitted) The curve C intersects the y-axis twice. Approximate the length of the curve between the two y-intercepts.

$x(t) = 0$

$2 - 3\cos t = 0$

$t = -0.841\ldots = A$

$t = 0.841\ldots = B$

$$\mathcal{L} = \int_A^B \sqrt{(3\sin t)^2 + (2\cos t)^2}\, dt = 3.756 \text{ or } 3.757$$

Multiple Choice:

C 14. A parametric curve is defined by $x = \sin t$ and $y = \csc t$ for $0 < t < \frac{\pi}{2}$. This curve is

(A) increasing & concave up (B) increasing & concave down (C) decreasing & concave up (D) decreasing & concave down (E) decreasing with a point of inflection

$x' = \cos t$, $y' = -\csc t \cot t$

$\frac{dy}{dx} = \frac{-\csc t \cot t}{\cos t}$
$= -\frac{1}{\sin t} \cdot \frac{\cos t}{\sin t} \cdot \frac{1}{\cos t}$
$= -(\sin t)^{-2}$
$= \frac{-1}{\sin^2 t}$

for $t \in (0, \frac{\pi}{2})$, $\frac{dy}{dx} < 0$
So curve is decreasing

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$
$= \frac{2(\sin t)^{-3}(\cos t)}{\cos t}$
$= \frac{2}{\sin^3 t}$

for $t \in (0, \frac{\pi}{2})$, $\frac{d^2y}{dx^2} > 0$
So, curve is concave up.

C 15. The parametric curve defined by $x = \ln t$, $y = t$ for $t > 0$ is identical to the graph of the function

(A) $y = \ln x$ for all real x (B) $y = \ln x$ for $x > 0$ (C) $y = e^x$ for all real x (D) $y = e^x$ for $x > 0$ (E) $y = \ln(e^x)$ for $x > 0$

$x = \ln t$, $y = t$, $t > 0$
eliminate parameter

$x = \ln y$, $y > 0$

So, $e^x = e^{\ln y}$, $y > 0$
$y = e^x$, $y > 0$ $\forall x \in \mathbb{R}$

E 16. The position of a particle in the xy-plane is given by $x = t^2 + 1$ and $y = \ln(2t+3)$ for all $t \geq 0$. The acceleration vector of the particle is

(A) $\left(2t, \frac{2}{2t+3}\right)$ (B) $\left(2t, -\frac{4}{(2t+3)^2}\right)$ (C) $\left(2, \frac{4}{(2t+3)^2}\right)$ (D) $\left(2, \frac{2}{(2t+3)^2}\right)$ (E) $\left(2, -\frac{4}{(2t+3)^2}\right)$

$\vec{s} = \langle t^2+1, \ln(2t+3) \rangle$
$\vec{s}\,' = \vec{v} = \langle 2t, \frac{2}{2t+3} \rangle = \langle 2t, 2(2t+3)^{-1} \rangle$
$\vec{s}\,'' = \vec{v}\,' = \vec{a} = \langle 2, -2(2t+3)^{-2}(2) \rangle$
$= \langle 2, \frac{-4}{(2t+3)^2} \rangle$