① (a) $P_4(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \approx \cos x = f(x) \quad c=0,\ x=0.8$

$\cos 0.8 \approx P_4(0.8) = 1 - \frac{(0.8)^2}{2!} + \frac{(0.8)^4}{4!} = 0.697 = A$

$R_4(0.8) = \left| \frac{f^{5}(z)}{5!}(0.8-0)^5 \right| \le \left| \frac{1}{5!}(0.8)^5 \right| = 0.0027306667 = B$

* $f^{(5)}(z)$ has a max value of one on the interval $[0, 0.8]$ since one is the amplitude of $\cos x$ and its derivatives

** The Lagrange error here is also the Alternating series error

(b) $\cos 0.8 \in [A-B, A+B] = [0.694336, 0.699797] = I$

* $\cos 0.8$ actually equals $0.6967067093 \in I$

(c) $\cos 0.8$ could equal 0.695 because $0.695 \in I$ from part (b).

② $f(x) = e^x \approx T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}, \quad c=0,\ x=-1$

(a) $f(-1) = e^{-1} \approx T_4(-1) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} = 0.375 = A$

$R_4(-1) = \left| \frac{f^5(z)}{5!}(0-(-1))^5 \right| \le \left| \frac{e}{5!} \right| = 0.0226523486 = B$

* $f^{(5)}(z)$ has a max value of e^1 on $|x| \le 1 \Rightarrow -1 \le x \le 1$

(b) $e^{-1} \in [A-B, A+B] = [0.352347, 0.397652] = I$

* e^{-1} actually equals $0.3678794412 \in I$

③ $f(5) = 6,\ f'(5) = 8,\ f''(5) = 30,\ f'''(5) = 48,\ \left| f^{(4)}(x) \right| \le 75 \ \forall x \in [5, 5.2]$

(a) $T_3(x) = 6 + 8(x-5) + \frac{30}{2!}(x-5)^2 + \frac{48}{3!}(x-5)^3 \approx f(x)$

(b) $f(5.2) \approx T_3(5.2) = 6 + 8(0.2) + 15(0.2)^2 + 8(0.2)^3 = \boxed{8.264} = A$

$R_3(5.2) = \left| \frac{f^{(4)}(z)}{4!}(5.2-5)^4 \right| \le \left| \frac{75}{4!}(0.2)^4 \right| = \boxed{0.005} = B$

(c) $f(5.2) \in [A-B, A+B] = \boxed{[8.259, 8.269]} = I$

(d) $f(5.2)$ could not equal 8.254 because $8.254 \notin I$ from part (c).

(4) $f(x) = \sin x$, $c = \frac{3\pi}{4}$

$f(x) = \sin x$, $f'\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = \cos x$, $f'\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f''(x) = -\sin x$, $f''\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f'''(x) = -\cos x$, $f'''\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f^{(4)}(x) = \sin x$, $f^{(4)}\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

So $f(x) = \sin x = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{3\pi}{4}\right) - \frac{\sqrt{2}/2}{2!}\left(x - \frac{3\pi}{4}\right)^2 + \frac{\sqrt{2}/2}{3!}\left(x - \frac{3\pi}{4}\right)^3 + \frac{\sqrt{2}/2}{4!}\left(x - \frac{3\pi}{4}\right)^4 + \cdots$

(5) (a) $f(x) = x\cos(x^3)$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots$$

$$\cos(x^3) = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots + \frac{(-1)^n x^{6n}}{(2n)!} + \cdots$$

$$f(x) = x\cos(x^3) = x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + \cdots + \frac{(-1)^n x^{6n+1}}{(2n)!} + \cdots$$

(b) $f(x) = \frac{1}{1+x^2}$

$$\begin{array}{r} 1 - x^2 + x^4 - x^6 + \cdots + \\ 1+x^2 \,\big)\, 1 \qquad\qquad \\ \underline{1 + x^2} \qquad\qquad \\ -x^2 \qquad\qquad \\ \underline{-x^2 - x^4} \qquad \\ x^4 \qquad \end{array}$$

So $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots$

(6) Radius and Interval of Convergence

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^n}{3^n n^2}$, $\lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{3^{n+1} \cdot (n+1)^2} \cdot \frac{3^n \cdot n^2}{(x-2)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{(x-2)\, n^2}{3(n+1)^2} \right| = \frac{1}{3}|x-2| < 1$

so $|x-2| < 3$, center $c = 2$

$\boxed{\text{Radius} = 3}$, Interval $[(-1, 5)]$

Test endpts:

$x = -1$: $\sum_{n=0}^{\infty} \frac{(-1)^n (-3)^n}{3^n n^2} = \sum_{n=0} \frac{(-1)^{2n} 3^n}{3^n n^2} = \sum_{n=0}^{\infty} \frac{1}{n^2} \rightarrow$ convergent p-series

$x = 5$: $\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{3^n n^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2} \rightarrow$ convergent alt series

$\boxed{\text{So Interval is } [-1, 5]}$

(b) $\sum_{n=0}^{\infty} (2n)!\,(x-5)^n$

$\lim_{n\to\infty} \left| \frac{(2n+2)!\,(x-5)^{n+1}}{(2n)!\,(x-5)^n} \right|$

$= \lim_{n\to\infty} \left| (2n+2)(2n+1)(x-5) \right|$

$= \infty \not< 1$ so

$\boxed{\text{Radius} = 0}$

$\boxed{\text{This series converges only at } x = 5 \text{, its center}}$

(7) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots+\frac{(-1)^n x^{2n}}{(2n)!}+\dots\right)}{x}$$

$$= \lim_{x\to 0} \frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}+\dots+\frac{(-1)^{n+1}x^{2n}}{(2n)!}+\dots}{x} = 0/0$$

$$= \lim_{x\to 0} \frac{\cancel{x}\left(\frac{x}{2!}-\frac{x^3}{4!}+\frac{x^5}{6!}+\dots+\frac{(-1)^{n+1}x^{2n-1}}{(2n)!}+\dots\right)}{\cancel{x}} = \boxed{0}$$

(8) $f^{(n)}(3) = \frac{(-1)^n n!}{5^n(n+3)}$, $f(3) = \frac{1}{3}$, $f'(3) = \frac{-1}{5\cdot 4}$, $f''(3) = \frac{2}{25\cdot 5}$, $f'''(3) = \frac{-6}{125\cdot 6}$, $f^{(4)}(3) = \frac{4!}{5^4\cdot 7}$

$\underline{c=3}$

(a) $T_4(x) = \frac{1}{3} - \frac{1}{20}(x-3) + \frac{2/125}{2!}(x-3)^2 - \frac{6/(6\cdot 125)}{3!}(x-3)^3 + \frac{4!/(5^4\cdot 7)}{4!}(x-3)^4 \approx f(x)$

$= \frac{1}{3} - \frac{1}{20}(x-3) + \frac{1}{125}(x-3)^2 - \frac{1}{750}(x-3)^3 + \frac{1}{4375}(x-3)^4$

(b) the n^{th} term for the Taylor series for $f(x)$ is:

$\frac{f^{(n)}(3)}{n!} = \frac{(-1)^n n!}{n!\, 5^n(n+3)} = \frac{(-1)^n}{5^n(n+3)}$. Radius: $\lim_{n\to\infty}\left|\frac{(x-3)^{n+1}}{5^{n+1}(n+4)}\cdot\frac{5^n(n+3)}{(x-3)^n}\right|$

$\lim_{n\to\infty}|x-3|\frac{n+3}{5(n+4)} = \frac{1}{5}|x-3| < 1$ so $|x-3|<5$ and $\boxed{\text{Radius} = 5}$

(c) $f(4) \approx T_4(4)$. $R_4(4) = \left|\frac{f^{(5)}(z)}{5!}(4-3)^5\right| \le \left|\frac{5!}{5!\,5^5\cdot 8}(1)^5\right| = \frac{1}{25000} = 0.00004 < \frac{1}{4000}$

$x=4, c=3$

* on the interval $[3,4]$, $f^{(5)}(z) \approx f^{(5)}(3) = \frac{(-1)5!}{5^5(8)}$

(9) $f(0)=1$, $f'(0)=\frac{1}{2}$, $f''(0) = -\frac{1}{4}$, $f'''(0) = \frac{3}{8}$, $|f^{(4)}(x)| \le 6\ \forall x \in (-1,1)$, $\underline{c=0}$

(a) $T_3(x) = 1 - \frac{1}{2}x - \frac{1/4}{2!}x^2 + \frac{3/8}{3!}x^3 \approx f(x)$

$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$

(b) $f(0.5) \approx T_3(0.5) = 0.7265625 = \frac{93}{128}$

(c) $R_3(0.5) = \left|\frac{f^{(4)}(z)}{4!}(0.5-0)^4\right| \le \left|\frac{6}{4!}\left(\frac{1}{2}\right)^4\right| = \underline{0.015625} = \underline{\frac{1}{64}}$

$c=0, x=0.5$

max possible error

(10) $f(x)=\sqrt{x}$

(a) $c=4$

$f(x)=x^{1/2},\ f(4)=2$

$f'(x)=\frac{1}{2}x^{-1/2},\ f'(4)=\frac{1}{4}$

$f''(x)=-\frac{1}{4}x^{-3/2},\ f''(x)=-\frac{1}{32}$

So $f(x)=\sqrt{x}\approx T_2(x)=2+\frac{1}{4}(x-4)-\frac{1/32}{2!}(x-4)^2$
$=2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2$

(b) $f(4.2)=\sqrt{4.2}\approx T_2(4.2)=2+\frac{1}{4}(0.2)-\frac{1}{64}(0.2)^2=2.049375=A$

(c) $T_2(4.2)=\left|\frac{f'''(z)}{3!}(4.2-4)^3\right|\leq\left|\frac{3}{3!(256)}(0.2)^3\right|=\frac{1}{(250)(256)}=0.000015625$

* $f'''(x)=\frac{3}{8}x^{-5/2}=\frac{3}{8\sqrt{x^5}}$. $f'''(x)$ has its max value on $[4,4.2]$ at $x=4$, so $f'''(z)=\frac{3}{8\sqrt{4^5}}=\frac{3}{256}$

or $\boxed{\frac{1}{64000}}$ MAX ERROR

(11) $f(x)=\sum_{n=0}^{\infty}\frac{x^n}{2^n}$

(a) Interval of Convergence: $\lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+1}}\cdot\frac{2^n}{x^n}\right|=\lim_{n\to\infty}\left|\frac{1}{2}x\right|=\frac{1}{2}|x|<1$

$|x-0|<2$

Radius $=2$

possible I.O.C. $[(-2,2)]$

Test endpts:

$x=-2:\ \sum_{n=0}^{\infty}\frac{(-2)^n}{2^n}=\sum_{n=0}^{\infty}(-1)^n\to$ Diverges

$x=2:\ \sum_{n=0}^{\infty}\frac{(2^n)}{2^n}=\sum_{n=0}^{\infty}1\to$ Diverges

So Interval of Convergence is $\boxed{(-2,2)}$

(b) $\sum_{n=0}^{\infty}\frac{x^n}{2^n}=1+\frac{x}{2}+\frac{x^2}{4}$

$f(x)\approx1+\frac{x}{2}+\frac{x^2}{4}=T_2(x)$

$f(-\frac{1}{2})\approx T_2(-\frac{1}{2})=1-\frac{1}{4}+\frac{1}{16}=\boxed{0.8125=\frac{13}{16}}$

(c) For $x=-\frac{1}{2}$, the series is an alternating series, so the maximum error will be the magnitude of the first unused term in the series for $f(-\frac{1}{2})=1-\frac{1}{4}+\frac{1}{16}-\underline{\frac{1}{64}}+\cdots+\frac{(-1)^n}{4^n}+\cdots$

1st unused term in $T_2(-\frac{1}{2})$

So error $\leq\left|-\frac{1}{64}\right|=\boxed{\frac{1}{64}}$

(12) $f(x) = \cos\left(3x + \frac{\pi}{6}\right) \quad c=0$

$f(x) = \cos\left(3x + \frac{\pi}{6}\right),\ f(0) = \frac{\sqrt{3}}{2}$

$f'(x) = -3\sin\left(3x + \frac{\pi}{6}\right),\ f'(0) = -\frac{3}{2}$

$f''(x) = -9\cos\left(3x + \frac{\pi}{6}\right),\ f''(0) = -\frac{9\sqrt{3}}{2}$

$f'''(x) = 27\sin\left(3x + \frac{\pi}{6}\right),\ f'''(0) = \frac{27}{2}$

$f^{(4)}(x) = 81\cos\left(3x + \frac{\pi}{6}\right),\ f^{(4)}(0) = \frac{81\sqrt{3}}{2}$

$f^{(5)}(x) = -243\sin\left(3x + \frac{\pi}{6}\right),\ f^{(5)}(0) = -\frac{243}{2}$

(a) $P_4(x) = \frac{\sqrt{3}}{2} - \frac{3}{2}x - \frac{9\sqrt{3}/2}{2!}x^2 + \frac{27/2}{3!}x^3 + \frac{81\sqrt{3}/2}{4!}x^4 \approx f(x)$

$P_4(x) = \frac{\sqrt{3}}{2} - \frac{3}{2}x - \frac{9\sqrt{3}}{4}x^2 + \frac{9}{4}x^3 + \frac{27\sqrt{3}}{16}x^4 \approx f(x)$

(b) $R_4\left(\frac{1}{6}\right) = \left|\frac{f^{(5)}(z)}{5!}\left(\frac{1}{6} - 0\right)^5\right| \leq \left|\frac{243}{5!}\left(\frac{1}{6}\right)^5\right| = \left(\frac{81}{40}\right)\left(\frac{1}{7776}\right) \approx \boxed{0.0002604 = A}$

$c=0,\ x=\frac{1}{6}$ * the max value of $|f^{(5)}(z)|$ is 243, the amplitude of $f^{(5)}(x)$

$\frac{1}{3000} \approx 0.0003333 = B$ $\boxed{A < B, \text{ where } A = \left|f\left(\tfrac{1}{6}\right) - P_4\left(\tfrac{1}{6}\right)\right|}$

(13) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots$

$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots$

$I = \int_0^1 e^{-x^2}\,dx = x - \frac{1}{3}x^3 + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)n!} + \cdots \Big|_0^1$

$= \left(1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} + \cdots + \frac{(-1)^n}{(2n+1)n!} + \cdots\right) - (0) = I$

* the approximation for I must be within $\frac{1}{1000}$ of the actual value.
* I is an alternating series, so error is less than the magnitude of the 1st unused term. The 1st term that is less than 0.001 is $\left|\frac{1}{11\cdot 5!}\right| \approx 0.000757$, so $I \approx 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!}$

$\approx \boxed{0.747486} = A$

* P.S. $\int_0^1 e^{-x^2}\,dx = 0.7468241328 = B$, so $|A - B| = 0.000006626 < 0.001$

actual error

(14) $c=5$, $f^{(n)}(5)=\dfrac{(-1)^n n!}{2^n(n+2)}$, $f(5)=\frac{1}{2}$

$f(6)\approx T_6(6)$, $R_6(6)=\left|\dfrac{f^{(7)}(z)}{7!}(6-5)^7\right|\le\left|\dfrac{5!}{7!\,2^5\cdot 7}\right|=\boxed{\dfrac{1}{(42)(32)(7)}=A}$

$*\ \left|f^{(7)}(z)\right|$ is approximated by $\left|f^{(7)}(5)\right|=\left|\dfrac{(-1)5!}{2^5(5+2)}\right|=\dfrac{5!}{2^5(7)}$

$\boxed{A=\dfrac{1}{9408}<\dfrac{1}{1000}}$

**notice the question did not ask us to approximate $f(6)$.

(15) $f(3)\approx T_4(3)$, $x=3, c=1$, $\left|f^{(5)}(x)\right|<0.01$

$R_4(3)=\left|\dfrac{f^{(5)}(z)}{5!}(3-1)^5\right|\le\left|\dfrac{0.01}{5!}(2^5)\right|=\boxed{0.0026666}$ $\boxed{E}$

(16) $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n!}$; Interval of Convergence: $\displaystyle\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\cdot\frac{n!}{x^n}\right|=\lim_{n\to\infty}\left|\frac{x}{n+1}\right|=0<1\ \forall x$

So Interval is $(-\infty,\infty)$ and radius is ∞. $\boxed{E}$
& all real x

(17) The coefficient of x^6 for $c=0$, $f(x)=\sin(x^2)$

$\sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+\cdots$, $\sin(x^2)=x^2-\dfrac{x^6}{3!}+\dfrac{x^{10}}{5!}+\cdots$

the coeff of x^6 is $-\dfrac{1}{3!}=\boxed{-\dfrac{1}{6}}$ $\boxed{A}$

(18) $\underbrace{1-\dfrac{1}{2!}+\dfrac{2}{3!}-\dfrac{3}{4!}+\dfrac{4}{5!}-\dfrac{5}{6!}}_{\text{1st 6 terms}}+\underbrace{\dfrac{6}{7!}}_{\text{1st unused term}}$ maximum error $\le\left|\dfrac{6}{7!}\right|=0.0011904762$

$\boxed{A}$

(19) $f'(x)=\sin(x^2)$, $\sin(x^2)=x-\dfrac{x^6}{3!}+\dfrac{x^{10}}{5!}+\cdots=f'(x)$,

$f(x)=\int f'(x)\,dx=C+\frac{1}{2}x^2-\dfrac{1}{7\cdot 3!}x^7+\dfrac{1}{11\cdot 5!}x^{10}+\cdots$; Coeff of x^7 is $-\dfrac{1}{7\cdot 3!}=\boxed{-\dfrac{1}{42}}$ $\boxed{D}$

(20) (A), (B), or (C) but not (D) since all this ENERGIZES Me!

— Best of Skill on the AP Exam