Déjà Vu, It's Algebra 2!

Lesson 10

Determinants & Matrix Multiplication

Every square matrix has a unique value associated with it called a determinant. We denote the determinant by using vertical bars, such as $\begin{vmatrix} 1 & 5 \\ 6 & 3 \end{vmatrix}$.

The determinant is a quite useful number, as you will soon see.

Definition of a determinant of a 2 x 2 matrix:

Verbal	Numeric	Algebraic
The determinant of a 2 by 2 matrix is the difference of the products of the diagonals.	$\det\begin{bmatrix} 1 & 5 \\ 6 & 3 \end{bmatrix} =$ $\begin{matrix} + \\ - \end{matrix}\begin{vmatrix} 1 & 5 \\ 6 & 3 \end{vmatrix} =$ $(1)(3)-(6)(5)=$ -27	$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} =$ $\begin{matrix} + \\ - \end{matrix}\begin{vmatrix} a & b \\ c & d \end{vmatrix} =$ $(a)(d)-(c)(b)=$

Note: the determinant of a matrix A may be denoted by either $\det A$ or $|A|$. Don't confuse the second with absolute value!!!

Example:

Find the determinant of the following matrix:

$$\begin{bmatrix} 4 & \frac{1}{3} \\ 6 & -\frac{1}{2} \end{bmatrix}$$

$$\begin{vmatrix} 4 & \frac{1}{3} \\ 6 & -\frac{1}{2} \end{vmatrix} = (4)\left(-\frac{1}{2}\right) - (6)\left(\frac{1}{3}\right) = -2 - 2 = -4$$

For a 3 x 3 square matrix, it gets a bit complicated. There are two methods: expansion by minors and diagonal products.

Example:

$$\text{Find } \det A = \begin{vmatrix} 1 & -2 & 3 \\ 4 & 2 & -1 \\ 6 & 3 & 7 \end{vmatrix}$$

$$\begin{aligned} &1(14+3) - -2(28+6) + 3(12-12) \\ &= 17 + 2(34) + 3(0) \\ &= 17 + 68 \\ &= 85 \end{aligned}$$

Determinants are useful in solving systems of equations by a method called *Cramer's Rule*.

Given a system of equations of the form:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$, the coefficient matrix is $$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

Let D be the determinant of the coefficient matrix:

$$D = \det\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

The solutions to the system are given by

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{D} \quad \text{and} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{D}$$

Example:

Solve the following system using Cramer's Rule.

$$\begin{cases} 2x = 10 - y \\ 8 = 3x - 2y \end{cases}$$

$\begin{cases} 2x + y = 10 \\ 3x - 2y = 8 \end{cases}$. The coefficient matrix is $\begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$. Its determinant is $D = \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = -4 - 3 = -7$

The solutions are:

$$x = \frac{\begin{vmatrix} 10 & 1 \\ 8 & -2 \end{vmatrix}}{-7} = \frac{-20-8}{-7} = \frac{-28}{-7} = 4$$

$$y = \frac{\begin{vmatrix} 2 & 10 \\ 3 & 8 \end{vmatrix}}{-7} = \frac{16-30}{-7} = \frac{-14}{-7} = 2$$

So the solution to the system (the point of intersection of the two lines) is the point $(4,2)$

This systematic process works equally well for linear systems of three variables, it just involves more steps.

Determinants are also useful in finding multiplicative inverses of a square matrix.

Remember, that is the multiplicative inverse matrix (or just the inverse matrix) is the matrix we may multiply by to get the multiplicative identity matrix. That is

$$AA^{-1} = I = A^{-1}A$$

The multiplicative identity matrix has 1s along the diagonal and 0s everywhere else.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition of the inverse of a 2 x 2 matrix:

The inverse of a 2 x 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $\det A = 0$, matrix A has no inverse, and is called a singular matrix.

Example:

Find the inverse M^{-1} of the following matrix, then verify it is so by finding MM^{-1}.

$$M = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

First find the determinant of *M*.
$\det M = 4 - 6 = -2$.

So $M^{-1} = \frac{1}{-2}\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}$

Verify:

$$MM^{-1} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} -2+3 & 6-6 \\ -1+1 & 3-2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Déjà RE-Vu

A very powerful method for solving systems of equations emerges when we combine matrix multiplication and inverse matrices, especially when we let the calculator handle the routine calculations for us.

Let's say we wanted to solve the following system of equations:

$$\begin{cases} 5x + 7y = 3z + 3 \\ 3x + 4y = 6 - 2z \\ x + 3y = 5z - 7 \end{cases}$$

Putting into standard form:

$\begin{cases} 5x + 7y - 3z = 3 \\ 3x + 4y + 2z = 6 \\ x + 3y - 5z = -7 \end{cases}$ We can create THREE matrices:

A = the coefficient matrix
B = variable matrix
C = constant matrix

$A = \begin{bmatrix} 5 & 7 & -3 \\ 3 & 4 & 2 \\ 1 & 3 & -5 \end{bmatrix}$, X= $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 6 \\ -7 \end{bmatrix}$

Using the variables of the matrices, we get the following equations:

$AX = B$

Solving for matrix X by multiplying BOTH sides by A^{-1} on the LEFT (remember, matrix multiplication is NOT commutative), we get

$A^{-1}AX = A^{-1}B$

$IX = A^{-1}B$

$X = A^{-1}B$

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[A]-1[B]
               [[4  ]
                [-2]
                [1  ]]
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Using a calculator, we get:

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}$, so $(x, y, z) = (4, -2, 1)$