Déjà Vu, It's Algebra 2!

Lesson 15

Polynomials: Factoring and Synthetic Division

The Remainder Theorem:

If a polynomial $P(x)$ is divided by $(x-a)$, then the remainder R is $P(a)$.

Example:

Divide $P(x)=x^4-2x^3+3x+1$ by $x-3$. What is the remainder?

By Long Division:

$$\begin{array}{r|l} & x^3+x^2+3x+12 \\ x-3 & x^4-2x^3+3x+1 \\ & \underline{x^4-3x^3} \\ & \quad x^3+3x+1 \\ & \quad \underline{x^3-3x^2} \\ & \qquad 3x^2+3x+1 \\ & \qquad \underline{3x^2-9x} \\ & \qquad\quad 12x+1 \\ & \qquad\quad \underline{12x-36} \\ & \qquad\qquad 37=R \end{array}$$

Or by Synthetic Division:

$$\begin{array}{c|ccccc} & 1 & -2 & 0 & 3 & 1 \\ [3] & & 3 & 3 & 9 & 36 \\ \hline & 1 & 1 & 3 & 12 & [37=R] \end{array}$$

By the Remainder Theorem:

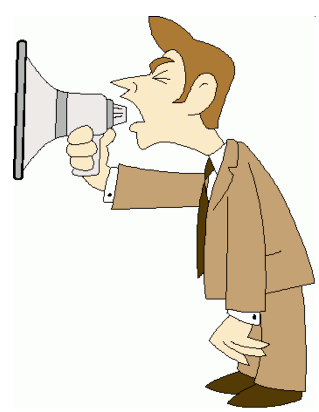
$R=P(3)=3^4-2\left(3^3\right)+3(3)+1=81-54+9+1=81-44=37$

Nested form a polynomial.

Example:

Write $P(x) = x^4 - 2x^3 + 3x + 1$ in nested form, then evaluate $P(3)$.

$$\left(x^3 - 2x^2 + 0x + 3\right)x + 1$$
$$= \left(\left(x^2 - 2x + 0\right)x + 3\right)x + 1$$
$$= \left(\left(\left(x - 2\right)x + 0\right)x + 3\right)x + 1$$
so
$$P(3) = \left(\left(\left(3 - 2\right)3 + 0\right)3 + 3\right)3 + 1$$
$$= 37$$



IMPORTANT RESULT

Synthetically dividing a polynomial by $x - a$ is equivalent to synthetically substituting with $x = a$. (Just remember to list coefficients in descending order with any necessary place holders for missing xs.

What if the remainder is zero?

Theorem:

If a polynomial $P(x)$ is divided by $x-a$ and the remainder is ZERO, then $x-a$ is a factor of $P(x)$.

This means, by the Remainder Theorem, that $P(a)=0$, or the GRAPH of $P(x)$ contains the point $(a,0)$.

The Factor Theorem:

A polynomial $P(x)$ has a linear factor $(x-a)$ IF AND ONLY IF $x=a$ is a root/zero/x-intercept of $P(x)$.

Why is this important?

If we know a root, we know a factor.
If we know a factor, we know a root!!

Why are roots important?

Roots of polynomial equations that model real-life behavior are the answers to the questions we are looking for!!!!!

Example:

Verify that $x = -3$ is a root/zero of the polynomial function $P(x) = 3x^5 + 18x^4 + 27x^3$ using synthetic division.

From the Factor theorem, we know that $x - -3 = x + 3$ is a factor. We can divide out $x + 3$ by synthetically substituting $x = -3$.

	3	18	27	0	0	0
[-3]		-9	-27	0	0	0
	3	9	0	0	0	[$0 = R$]

We could also try to find roots by factoring the polynomial. If we can factor it completely, we'll be staring at all of our linear factors, which means we'll implicitly be staring at all or our solutions!

Example:

$$P(x) = 3x^5 + 18x^4 + 27x^3 = 0$$

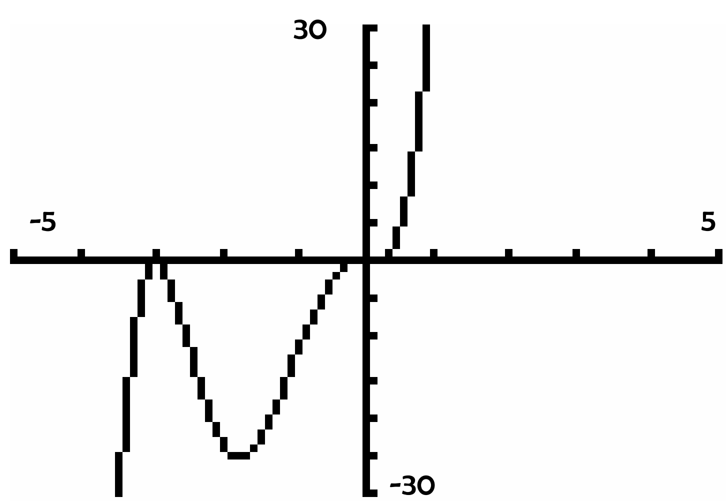
$$3x^3\left(x^2 + 6x + 9\right) = 0$$

$$3(x-0)^3(x+3)^2 = 0$$

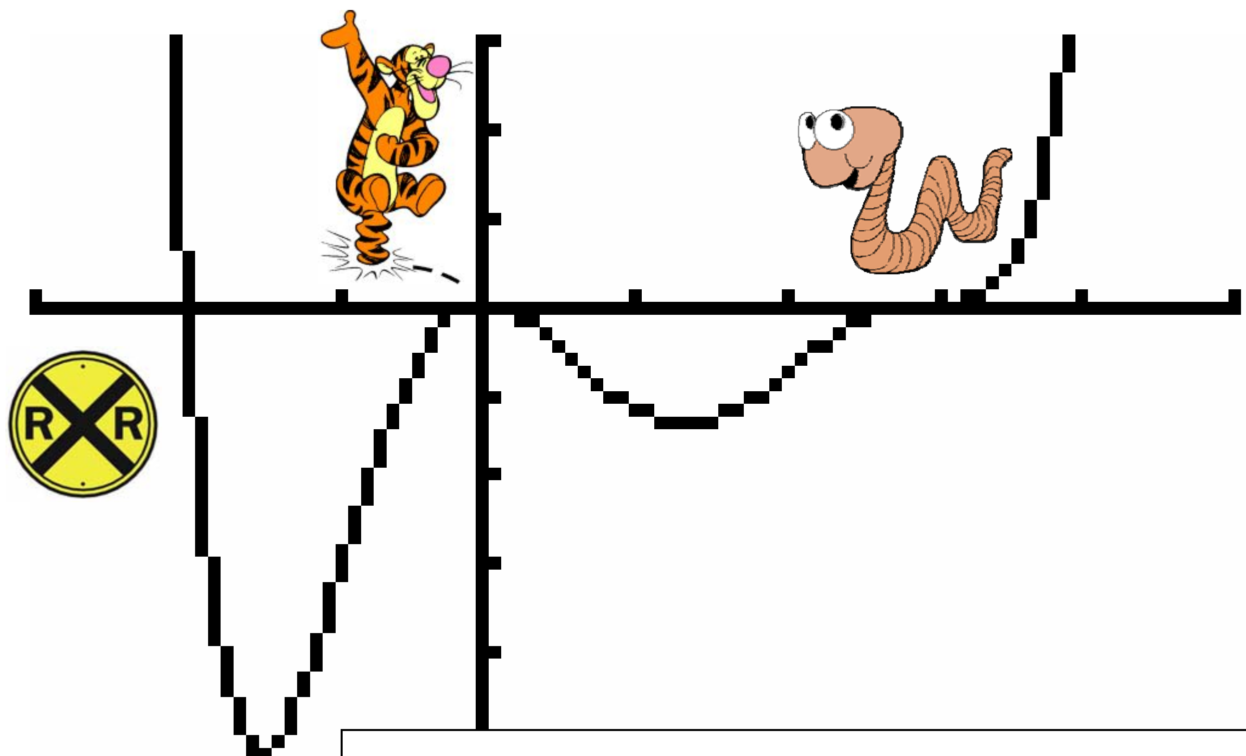
$$3(x-0)(x-0)(x-0)(x+3)(x+3) = 0$$

$$x = 0,\ 0,\ 0,\ -3,\ -3$$

$$x = 0\ (\text{m}3), -3\ (m2)$$



We know how that the **factors in the equation give us the x-intercepts of the graph**, but now we've made the connection that the **exponents on the factors are the multiplicities** of the roots. There are three distinct types: **Crossings, Bounces, and Inflections**.



$$P(x) = x^6 - 7x^5 + 9x^4 + 27x^3 - 54x^2$$

What does the Factor Theorem tell us about $P(x)$?

$P(x) = A(x+2)^1(x-0)^2(x-3)^3$

$P(x) = Ax^2(x+2)(x-3)^3$

where A is an unknown vertical dilation factor, $A \neq 0$

Déjà RE-Vu

Synthetically divide

$$P(x) = x^6 - 7x^5 + 9x^4 + 27x^3 - 54x^2$$

by all of its factors, including multiplicities.

	1	−7	9	27	−54	0	0
[−2]		-2	18	-54	54	0	0
	1	-9	27	-27	0	0	[0]
[0]		0	0	0	0	0	
	1	-9	27	-27	0	[0]	
[0]		0	0	0	0		
	1	-9	27	-27	[0]		
[3]		3	-18	27			
	1	-6	9	[0]			
[3]		3	-9				
	1	-3	[0]				
[3]		3					
	1	[0]					

References:
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