Déjà Vu, It's Algebra 2!

Lesson 32

Permutations & Combinations

Permutations

Let's say we want to arrange the three letters of the word **DOG** into two-letter groups where **OG** is different from **GO** and no letters are repeated.

The easiest way to figure this out is to simply list them as follows:

DO DG OG OD GO GD (6 ways)

In this case, since order matters, we are finding the number of *permutations* of size 2 that can be taken from a set of size 3.

We often write this as either ${}_nP_r$ or $P(n,r)$

In our case of the word **DOG**, we'd write it as ${}_3P_2$ or $P(3,2)$

Listing and counting the possibilities is a great method . . . if the list is small, but what if we want to find all the 4-letter permutations (without repeat) of the word **CUNEIFORMS**, a 10-letter word?!?

What is ${}_{10}P_4$?

Rather than list them, we can use the Fundamental Counting Principal.

There are 10 possibilities for the first letter, 9 for the second, 8 for the third, and 7 for the last letter. We can find the total number of 4-letter permutations by multiplying $10 \times 9 \times 8 \times 7 = 5040$.

The pattern above is *part* of what's called a factorial.

The factorial function (symbol: !) just means to multiply a series of descending natural numbers.

Examples:

$4! = 4 \times 3 \times 2 \times 1 = 24$

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

$1! = 1$

Note: it is generally agreed that $0! = 1$. It may seem funny that multiplying no numbers together gets you 1, but it helps simplify a lot of equations.

To obtain only the part 10 x 9 x 8 x 7, we need to divide 10! (because there are 10 letters total), by $(10-4)! = 6!$ (subtracting from the total number of letters the number of objects we are choosing in each permutation.) It looks like this . . .

$$_{10}P_4 = \frac{10!}{(10-4)!} = \frac{10!}{6!} = \frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{6\times5\times4\times3\times2\times1}$$

$$= 10\times9\times8\times7 = 5040$$

From this example, we can obtain the general formula for finding the number of permutations or size *r*, without repetition, taken from *n* objects.

$${}_nP_r = P(n,r) = \frac{n!}{(n-r)!}$$

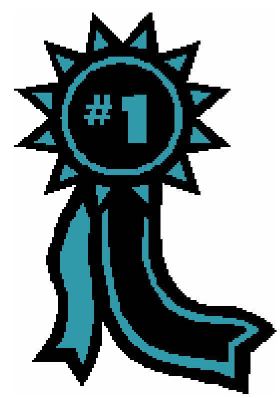
(NO REPEATS, ORDER MATTERS)

For our **DOG** example, this was

$${}_3P_2 = \frac{3!}{(3-2)!} = \frac{6}{1} = 6$$

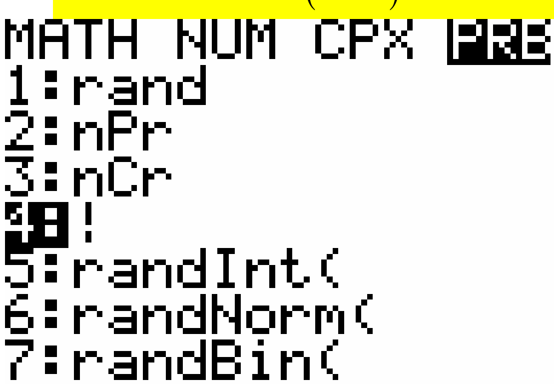
Example:

How many different ways can first, second, and third places be awarded to a group of 15 contenders?



This is a permutation where order definitely matters, and repeats are out of the question.

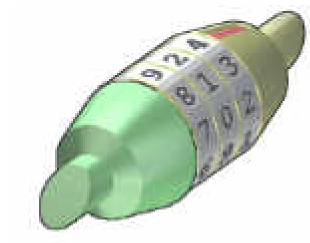
$${}_{15}P_3 = P(15,3) = \frac{15!}{(15-3)!} = \frac{15!}{12!} = 15 \times 14 \times 13 = 2730$$



Note: There is a factorial button on your calculator, found under “MATH, PRB, 4.”

Example:

If you were trying to crack a combination lock with 4 sliders, each numbered 0-9, how many attempts would you have to try before you were guaranteed to open it?



This IS a permutation (order matters, a combination of 1234 is different from 4321), but repeats ARE allowed, meaning the combination to the lock could be 3333. In this case, we don't need our fancy formula, but rather use the Fundamental Counting Principal.
There are 10 possibilities for the first number, 10 for the second, 10 for the third, and 10 for the last. This means there are 10x10x10x10= $10^4 = 10{,}000$ possible lock combinations. Good luck trying them all!

COMBINATIONS

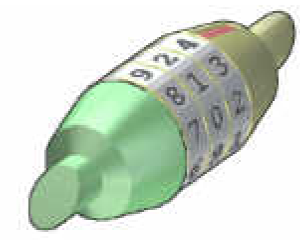
In English, we use the word "combination" loosely without thinking if the order of things is important. For instance:

- *"My diet consists of a combination of fruits, vegetables, dairy, meat, and math."* We don't care what the order the things in my diet are. I could have said "vegetables, dairy, math, meat, and fruits," or "math, veggies, meat, dairy, fruit, and meat."

- *"The combination to my safe is 1496"* Of course, now we DO care about the order, especially if you want to steal my stuff. A combination of 9461 would NOT open the safe.

In Mathematics, we have a much more precise and formal language.

- If the order does NOT matter, it is a Combination.
- If the order DOES matter, it is a Permutation.



So we really should call this a Permutation Lock!

Think of a Permutation as an ordered Combination, where Position matters.

Let's return to the DOG example.
If we wanted to find the number of Combinations of size 2 without repeated letters that can be made from the 3 letters in DOG, order doesn't matter. OG is the same as GO. We can list the combinations:

DO DG OG

We can say "3 choose 2" and write it in any one of the three ways:

$$_3C_2 = C(3,2) = \binom{3}{2}$$

What if we want to find the number of 4-letter combinations of the aforementioned 10-letter word **CUNEIFORMS**? Again, we don't want to have to write all the combos out.

Since we already know that ${}_{10}P_4 = 1050$, we can use this information to find ${}_{10}C_4$. The number of combinations should be much smaller than the number of permutations. This is because while arrangements of **CUNE, CUEN, CNUE, NUEC**, ... are unique permutations, they are the SAME combination!! We must then "reduce" our number of 5040 by the number of permutations of these 4-letter arrangements, $4 \times 3 \times 2 \times 1 = 4!$

The formula then becomes . . .

$$ {}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!\,r!} $$

For our **CUNEIFORMS** example,

$$ {}_{10}C_4 = \binom{10}{4} = \frac{10!}{(10-4)!\cdot 4!} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210 $$

To verify our DOG example,

$${}_3C_2 = \binom{3}{2} = \frac{3!}{(3-2)!2!} = \frac{6}{2} = 3$$

Example:

A lottery game pays the jackpot if an individual correctly chooses the 6

winning numbers from a field of 50. What are his chances of winning?

This is a combination without repeats. The order does not matter, but each number is only drawn at most once. To find the probability of winning, we first need to find the number of combinations of 50 choose 5.

${}_{50}C_6 = \binom{50}{6} = \frac{50!}{(50-6)!6!} = \frac{50!}{44!6!} = 15,890,700$. His probability of selecting the 1 correct combination is $\frac{1}{15890700} \times 100 = 0.0000063\%$. The odds of being struck by lightning are estimated to be 1/700,000. This means you are almost 23 more times likely to get struck by lightning than win the lottery, although is much more fun to try to win the lottery.

Déjà RE-Vu

How many unique permutations of the word MISSISSIPPI are there?

This is a permutation with indistinguishable objects, namely the repeated letters of I, S, and P. No fear! In much the same spirit as deriving our combination formula, we'll do the same for this: we simply need to divide out the number of permutations of the repeated letter that are the same!

Mississippi has 11 letters. Here's the frequency of each letter. 1 M, 4 I, 4 S, 2 P. The number of arrangements is thus

$=\frac{11!}{1!4!4!2!}=34,650$. Notice how this would differ from the permutations of an 11-letter word with all different letters, such as FORTUNATELY, which would be 11! = 39,916,800.

Math is everywhere!

References:
http://mathforum.org/dr.math/faq/faq.comb.perm.html
http://www.mathsisfun.com/combinatorics/combinations-permutations.html
http://www.locallender.info/images/states/mississippi.gif
http://www.medicine.uiowa.edu/cigw/image_cartoon/dog.gif
http://www.frontiernet.net/~mblow/images/thematic%20images/cuneiform.gif
http://www.ci.gresham.or.us/departments/ocm/gallery/juried_show_2006/graphics/ribbon.gif
http://www.winningwithnumbers.com/lottery/games/lottery.gif