Déjà Vu, It's Algebra 2!

Lesson 33

A look at Symmetry and Groups

In geometry, we studied some visual patterns of the world around us, ranging from obvious mathematical patterns (such as circles, squares, and pyramids) to those found in nature (such as growth patterns of plants or stripes on an animal.)

In algebra, we studied "other" patterns, not so much of shape, but rather of form, such as symmetry. The only way to study the perception of symmetry is through the language of mathematics. But how is this done?

We must first find a precise way of looking at symmetry, one that allows us to use formal reasoning leading to equations and formulas. This, in fact, is the most difficult step, requiring great insight, patience, and creativity.

So what IS symmetry? A casual definition would be to say an object (a vase, a flower) is symmetrical if it looks the same from different angle. That is, after the object is manipulated is some way, does it look the same as it did before? Does it have the position, size, shape, or orientation?

For example, take a circle and rotate it about its center through ANY angle and it looks EXACTLY the same as before. We can say that a circle is symmetrical for any rotation about its center.

The circle also has symmetry for any reflection across any diameter. In fact, the circle is special because it has many symmetries.

The square, however, has less symmetry than a circle. If we rotate a square through 90° or a multiple of 90°, it looks the same. However, if we rotate it through, say, 45° it looks different. Other symmetries of the square are reflecting it across either of the two diagonals or the two lined parallel to the edge passing through the center.

As with the circle, the symmetries of the square are those for which the manipulations leave it looking EXACTLY the same in position, shape, and orientation.

My face has symmetry, although not as much as the square. A vertical line through the middle of my nose has the left and right side looking the same, for the most part. We must assume the ideal versions of real-life objects for study.

Extend the line through my entire body, and you see that in 3-dimensions, the human body has symmetry. This is why it appears that our reflections in mirrors “flip” from left to right. It only appears that way because of our vertical symmetry. The mirror itself doesn’t flip anything (try laying down on your side in front of a mirror if you don’t believe me.)

In general, the more manipulations that leave an object unchanged, the greater its symmetry. Mathematically, these manipulations are called transformations, and they leave the object invariant.

To study symmetry mathematically, we must force ourselves to look at the transformations of the objects rather than the objects themselves.

It turns out that there is an "arithmetic" of these transformations, just as there is an arithmetic for numbers. Just like adding or multiplying two numbers yields a new number, in the "arithmetic of transformations" a sequence of transformations produces another transformation.

Both arithmetics work in a similar fashion, but with noticeable differences. The study of the "arithmetic of transformations" is called "group theory."

Relying on the *patterns* we notice in combining numbers, in group theory, we are interested in the emerging *patterns* of combining symmetries (essentially going from patterns, to patterns of patterns, to patterns of patterns of patterns.)

For any figure, the symmetry group of that figure is the collection of all invariant transformations. For example, the symmetry group for the circle consists of all rotations, reflections in any diameter, and any combination of the two.

Let's explore this new arithmetic of transformations with the symmetry group of the circle.

Instead of the traditional *x, y,* and *z,* we use capital letters, such as *S, T,* and *W.*

Let S and T be two transformations of the circle's symmetry group. Applying S and then T, we get another member of the symmetry group. We denote this using the composition notation:

$$T \circ S$$

Looking for patterns, mathematicians naturally see which properties of whole numbers are mirrored in combining symmetry transformations.

1. The operation is *associative*:

$$\left(T \circ S\right) \circ W = T \circ \left(S \circ W\right)$$

2. There is an *identity* transformation, I:

$$T \circ I = I \circ T = T$$

3. Every transformation has an *inverse*:

$$T \circ S = S \circ T = I$$

Although we were looking at symmetries of the circle, those properties hold true for the symmetry group of ANY figure, and they mirror the properties of addition of whole numbers.

With abstraction being the most powerful tool in the mathematician's arsenal, the next step is to abandon the actual symmetries and the objects themselves and focus strictly on the arithmetic. The result is the completely abstract definition of the mathematical object called the "group."

In general, if we have some set, say G, of entities (be them symmetry transformations of some figure or not), and some operation that combines any two elements x and y in the set G to give a further element $x * y$ in G, we call the collection a *group* if the following three conditions are met:

- G1. For all x, y, z, in G, $(x * y) * z = x * (y * z)$
- G2. There is an element e in G such that $x * e = e * x = x$ for all x in G. e is called the identity element.
- G3. For each element x in G, there is an element y in G such that $x * y = y * x = e$, where e is the identity element.

These three conditions (or axioms for a group) are simply the properties of associativity, identity, and inverses.

So the collection of all symmetry transformations of a figure is a group, where *G* is the collection of all symmetry transformations of the figure under the operation of combining two symmetry transformations.

Additionally, if *G* represents the set of whole numbers under the operation of addition, the resulting structure is a group. This is not true for multiplication, unless you consider the set of non-zero rational numbers.

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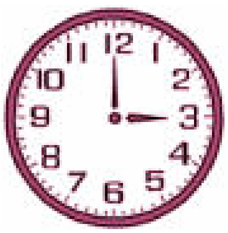

The way we tell time is another example of a group. In clock arithmetic, the 12-hour clock has the whole numbers 1 to 12 (which constitute the set *G*), and we add them according to the rule that we return to 1 when we go past 12. For example, $9+6=3$, which at first glances looks strange until we remember

9 o'clock + 6 hours = 3 o'clock

Let's check the three axioms for groups:

☙G1: (9 o'clock + 6 hours) + 2 hours = 9 o'clock + (6 hours + 2 hours) = 5 o'clock.

$$\text{So, } (9+6)+2=9+(6+2)$$

☙G2: In clock arithmetic, adding 12 takes us right back to the same hour. 2 o'clock + 12 hours = 12 hours + 2 o'clock = 2 o'clock.

$$\text{So, } 2+12=12+2=2$$

☙G3: To get the inverse in clock arithmetic, just travel around the clock until you get to 12. 7 o'clock + 5 hours = 5 hours + 7 o'clock = 12 o'clock.

$$\text{So, } 7+5=5+7=12$$

Although clock groups are amusing, they're not of particular interest to mathematicians. The example DOES illustrate, though, how the concept of group can arise in many different contexts and can be found hiding in all sorts of phenomena. In fact, groups are the building blocks of many other algebraic structures. Whether you find this new algebra hard or easy, it does illustrate the incredible power and utility of abstraction in mathematics.

If tomorrow you wake up and discover a novel mathematical structure and determine it is a group, you know immediately that that structure has a unique identity and a unique inverse, simply based on the 3 axioms of a group. The cost of this increased efficiency is that you must learn to work with highly abstract structures—with abstract patterns of abstract entities. You have to live inside your own imagination.

Anyone can train themselves to think abstractly. It's what we do when we learn a new language. However, some have developed an expectation that they cannot do so. And as with most things in life, we tend to find what we expect.

Math is everywhere!

References:
http://math.arizona.edu/~emcnicho/School_Page/M105/Handouts/Modular.htm
The Math Gene, by Keith Devlin. Basic Books. 2000.