Tower of Hanoi

Fascinating Facts

The Tower of Hanoi (sometimes referred to as the Tower of Brahma or the End of the World Puzzle) was invented by the French mathematician, Edouard Lucas, in 1883. He was inspired by a legend that tells of a Hindu temple where the pyramid puzzle might have been used for the mental discipline of young priests. Legend says that at the beginning of time the priests in the temple were given a stack of 64 gold disks, each one a little smaller than the one beneath it. Their assignment was to transfer the 64 disks from one of the three poles to another, with one important provision: a large disk could never be placed on top of a smaller one. The priests worked very efficiently, day and night. When they finished their work, the myth said, the temple would crumble into dust and the world would vanish.

Where's the Math in this Game?

The number of separate transfers of single disks the priests must make to transfer the tower is $2^{64}-1$, or 18,446,744,073,709,551,615 (that's 18 quintillion +) moves! If the priests worked day and night, making one move every second, it would take slightly more than 580 billion years to accomplish the job! You have a great deal fewer disks than 64 here.

Can you calculate the number of moves it will take you to move the disks from one of the three poles to another?

<u>Discovering the Mathematics Behind the Game</u>

If you've played the game as well as possible, you should have discovered that the minimum number of turns it takes to win (for 1, 2, 3, and 4 disks, respectively) are 1, 3, 7, and 15.

Looking for the Pattern

To figure out how many turns it'll take for more than four disks, and to figure out how long it'll take the monks to finish their task, you need to find a pattern relating the number of disks to the minimum number of turns it takes to win the game.

Write down the minimum number of turns it takes for one, two, three, and four disks. Can you conjecture a pattern? (Hint: try adding one to each of the numbers; do you recognize a pattern now?)

Justifying the Pattern

It's great that you've found a pattern. But maybe the pattern is just a coincidence, and doesn't continue for larger numbers of disks. To see if it does, we need a more systematic, mathematical approach. We will proceed by ***<u>induction</u>***, not unlike a mathematical domino effect: if we know how many turns it takes to win the n-disk game, we'll try to use that to find how many turns it takes to win the $(n+1)$-disk game. Having done that, we'll be able to find the number of turns for any n.

So, suppose we know that it takes M turns to move a pile of n disks from one post to another. How many turns will it take to move $n+1$ disks?

Think of it this way: *imagine you can move the top n disks as a single unit*. Now, what's the quickest way to move all $n+1$ disks? Your answer should involve a certain number of moves of the top n disks (as a single unit), and a certain number of moves of the bottom disk (by itself).

Now, although this is illegal (you can't move the top *n* disks as a unit), *you can achieve the same effect in M legal moves*, because we know that it takes *M* turns to move a pile of *n* disks from one post to another.

Therefore, you now have a procedure for legally moving all *n*+1 disks to their new home: follow the procedure you devised before, but instead of moving the top *n* disks as a unit, use the *M* legal moves required to achieve the same effect. How many turns does this procedure take?

<u>Discovering the Mathematics, Continued</u>

You should have realized that, to move all $n+1$ disks from post 1 to post 3, you can start by moving the top *n* disks to post 2. Now you're free to move the bottom disk from post 1 to post 3. Finally, you can move the top *n* disks from post 2 to post 3 (back on top of the the bottom disk). This takes *M* turns for the first stage, 1 turn for the second stage, and *M* turns for the third stage: a total of $2M+1$ turns.

In other words: if $a(n)$ is the minimum number of turns it takes to move *n* disks from one post to another, then $a(n+1)=2\cdot a(n)+1$.

HOLD ON A DISK-STACKING MINUTE!!! This isn't quite right. It just means you can move $n+1$ disks in $2\cdot a(n)+1$ turns.

But, is THIS the minimum number of turns, or can you do it in fewer?

See if you can prove that it really is the minimum number of turns. (Hint: the bottom disk will have to be moved at some point. What must happen before the bottom disk can be moved? What's the smallest number of turns this could possibly take? What must happen after the bottom disk is moved? What's the smallest number of turns this could possibly take?)

Tying this in to the pattern.

Our formula $a(n+1)=2\cdot a(n)+1$ is great, but the pattern we want is something that says what $a(n)$ is in terms of *n*. You should have conjectured that $a(n)=f(n)$ where $f(n)$ is some explicit function of the *n* that you wrote down. *How can you tell if your conjecture is right or not?*

Note: If you couldn't conjecture a formula for $f(n)$ before, you may be able to do so now, because now you have more numbers to work with! Before we knew that the number of turns required for 1, 2, 3, and 4 disks were 1, 3, 7, and 15, but now we can also figure out the number of turns for 5 disks ($(2)(15)+1=31$), the number of turns for 6 disks ($(2)(31)+1=63$), and so on. This is called a **<u>RECURSIVE RELATION</u>**, because it relies on generating numbers in progressive order, each calculation depending on the previous value(s).

Can you see a pattern to the numbers 1, 3, 7, 15, 31, 63, . . . ? Try adding 1 to them all. Now you should be able to make a conjecture. How can you tell if it's right or not?

The answer: Proof by Induction

1. Show that it's true for $n=1$, that is, show that $a(1)=f(1)$. This is called the **<u>BASIS</u>**!
2. Show that, if it's true for $n=k$, for some positive integer k, then it's also true for $n=k+1$. That is, assume that $a(k)=f(k)$ for some positive integer k.
3. Attempt to show that it's true in the $k+1$ case. That is, $a(k+1)=f(k+1)$. Like dominos (the playing tiles, not the pizza company . . . . mmmmmmm, pizza!!)

You can accomplish step 3 by using the fact that $a(k+1) = 2 \cdot a(k) + 1 = 2 \cdot f(k) + 1$, so you just have to make sure that $2 \cdot f(k) + 1$ is the same as $f(k+1)$. This you can do just by using your formula for what $f(k)$ is, and checking that $2 \cdot f(k) + 1 = f(k+1)$.

(If you were unable to make a conjecture for what the formula should be, the answer is given in the next section. But the task of proving that it's true is left up to you!)

Back to the Legend

Now we know the minimum number of turns required to move *n* disks. If the monks work fast and move one disk every second, how many years will it take them to finish the job? Are you surprised by your answer? Do you think the chilling prophecy is likely to be true?

The Answer

The minimum number of moves it takes to move *n* disks is $2 \cdot a(n) + 1$.

Here's my derivation of the recursion formula and the leap to the formula above.—Korpi

Let H_n be the minimum number of moves required to move n disks from one peg to the other following the rules of the game.

For a, $H_1 = 1$ or written in terms of n: $H_n = 1$

To move two discs, it will require all the moves required to move the previous number of disks, plus one more move to relocate the bottom disk, then it will again require all the moves from the previous number of disks to restack them on top the now relocated bottom disk. This means twice the previous moves plus one, or . . .

For $n = 2$, $H_2 = 2H_1 + 1 = 2(1) + 1 = 3$ or again in terms of n: $H_n = 2H_{n-1} + 1$

For $n = 3$, $H_2 = 2H_1 + 1 = 2(3) + 1 = 7$ or again in terms of n: $H_n = 2H_{n-1} + 1$

We can continue this pattern indefinitely in terms of n, all it requires is that we have an initial condition for $n = 1$.

So, we can represent all cases of *n* by a simple recursive relation:

$H_n = 2H_{n-1} + 1,\ H_1 = 1$ (this is called the **INCUCTIVE HYPOTHESIS**)

Here's a table of the required moves for different values of n,

n	H_n
1	1
2	$2(1)+1=3$
3	$2(3)+1=7$
4	$2(7)+1=15$
5	$2(15)+1=31$
6	$2(31)+1=63$
7	$2(63)+1=127$
$\vdots$	$\vdots$
n	$2H_{n-1}+1$

RECURSIVE RELATIONS are nice, but because they rely on generating all the previous values, computing values for large values of n can be both tedious and cumbersome. What would be BETTER, is if we can derive an EXPLICIT formula that would generate the n^{th} value directly by direct subsititution. That is, not H_n, but rather $H(n)$. From this chart, you might be able to discern the formula, especially with the hints given earlier, but there is another, more mathematically rigid way:

Assume the basis and inductive hypothesis from above are true. So,

$$\begin{aligned}
H_n &= 2H_{n-1}+1 \\
&= 2(2H_{n-2}+1)+1 = 2^2(H_{n-2})+2+1 \\
&= 2^2(2H_{n-3}+1)+2+1 = 2^3(H_{n-3})+2^2+2^1+2^0 \qquad \text{(notice } 2=2^1 \text{ and } 1=2^0\text{)} \\
&= 2^3(2H_{n-4}+1)+2^2+2^1+2^0 = 2^4(H_{n-4})+2^3+2^2+2^1+2^0 \\
&\vdots
\end{aligned}$$

This pattern will continue until $H_{n-i}=H_1=1$, giving the final line of

$$H_n = 2^{n-1}+2^{n-2}+\cdots+2^3+2^2+2^1+2^0$$

rewriting the terms in increasing powers of 2:

$$H_n = 2^0+2^1+2^2+2^3+\cdots+2^{n-2}+2^{n-1} = \sum_{i=1}^{n} 2^{n-1}$$

The above is known as a SERIES (sum of terms generated by a rule). More specifically, because each term increases by a constant factor of 2, this is called a **GEOMETRIC SERIES**, where 2 is called r, the **RATIO**. Even more speocifically, because n is a finite number, this is called a **FINITE GEOMETRIC SERIES**. There is a well-known formula for finding the sum of a finite geometric series as a function of both n , r, and a_1 (the first term of the series). The formula for the sum fo the first n terms, S_n, is:

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

"But, from where did this awesome formula come?!" you incredulously ask in a grammatically correct manner. Let's take a bit of a detour and go off on a tangent (bad calculus joke).

First, let's express the geometric series more generally as

$$S_n = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + a_1 r^5 + \cdots + a_1 r^{n-1}$$

Now, we multiply both sides by the ratio, r

$$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + a_1 r^5 + a_1 r^6 + \cdots + a_1 r^{n}$$

Now, we subtract these two equations, top minus bottom, and what so many terms just disappear! Doing this, we arrive at

$$\begin{aligned} S_n &= a_1 + a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + a_1 r^5 + \cdots + a_1 r^{n-1} \\ -rS_n &= \qquad a_1 r + a_1 r^2 + a_1 r^3 + a_1 r^4 + a_1 r^5 + a_1 r^6 + \cdots + a_1 r^{n} \\ \hline S_n - rS_n &= a_1 - a_1 r^n \\ S_n(1-r) &= a_1(1-r^n) \end{aligned}$$

$$S_n = \frac{a_1(1-r^n)}{1-r}$$

Now back to our quest to find the explict formula as a function of n. Using the above formula with $a_1 = 1$ and $r = 2$:

$$S_{64} = 2^{64} - 1$$

So,

$$S_n = 2^n - 1$$

Now back to the Hindu priests and their 64 golden disks.

By this formula, which we now indubitably know gives us the fewest number of moves required to move n disks, we arrive at the total number of moves required by the priests to be

$$\begin{aligned} S_{64} &= 2^{64} - 1 \\ S_{64} &= 1.8446744073709551616 \times 10^{19} - 1 \\ S_{64} &= 18{,}446{,}744{,}073{,}709{,}551{,}616 - 1 \\ S_{64} &= 18{,}446{,}744{,}073{,}709{,}551{,}615 \end{aligned}$$

If the priests worked day and night, making one move every second, it would take slightly more than 580 billion years to accomplish the job!

If the legend is true, it looks like the Hindu priests' temple and Planet Earth are safe for a few more years!