§8.3—Sequences & Series: Convergence & Divergence

A sequence is simply list of things generated by a rule

More formally, a **sequence** is a function whose domain is the set of positive integers, or **natural numbers**, *n*, such that $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. The range of the function are called the terms in the sequence,

$$a_1, a_2, a_3, \ldots, a_{n,\ldots}$$

Where a_n is called the ***n*th term** (or rule of sequence), and we denote the sequence by $\{a_n\}$.

The sequence can be expressed by either
1) an ample number of terms in the sequence, separated by commas
2) an explicit function defined by the **rule of sequence**
3) the rule of sequence set off in braces.

Example 1:
The sequence $2, 4, 6, 8, \ldots$ is the sequence of even numbers. Express the same sequence as a rule of a non-negative integer *n*. The sequence $1, 3, 5, \ldots$ is the sequence of odd numbers. Express the same sequence as a rule of a non-negative integer *n*. How many in the list are needed to establish the "rule" in the absence of the explicitly-stated rule?

***NOTE: When given a sequence as a list, the first term is usually designated to be associated with $n = 1$. This is because we are using *n* as an ordinal (or counting) number, rather than a cardinal number.

We will be primarily interested in what happens to the sequence for increasingly large values of *n*.

Example 2:
If $a_n = \left\{ \dfrac{4n}{3 + 2n} \right\}$, list out the first five terms, then estimate $\lim\limits_{n \to \infty} a_n$.

FACT:

Let $\{a_n\}$ be a sequence of real numbers.

Possibilities:

1) If $\lim_{n\to\infty} a_n = \infty$, then $\{a_n\}$ diverges to infinity
2) If $\lim_{n\to\infty} a_n = -\infty$, then $\{a_n\}$ diverges to negative infinity
3) If $\lim_{n\to\infty} a_n = c$, an finite real number, then $\{a_n\}$ converges to c
4) If $\lim_{n\to\infty} a_n$ oscillates between two fixed numbers, then $\{a_n\}$ diverges by oscillation

Definition:

$n!$ is read as "n factorial." It is defined recursively as $n! = n(n-1)!$ or as

$$n! = n(n-1)! = n(n-1)(n-2)(n-3)\cdots 3\cdot 2\cdot 1$$

Por ejemplo: $9! = 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$

Example 3:

Determine whether the following sequences converge or diverge.

(a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$

(b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$

(c) $a_n = 3 + (-1)^n$

(d) $a_n = \frac{n}{1-2n}$

(e) $a_n = \frac{\ln n}{n}$

(f) $a_n = \frac{n!}{(n+2)!}$

(g) $a_n = \frac{2n!}{(n-1)!}$

(h) $a_n = \frac{n+(-1)^n}{n}$

(i) $a_n = \frac{(-1)^n (n-1)}{n}$

(j) $a_n = \frac{2^n}{(n+1)!}$

(k) $a_n = \left(1 + \frac{1}{n}\right)^n$

(l) $\left\{\frac{(2n)!}{n^n}\right\}$

Sometimes, albeit rarely, we have to write the rule of sequence as a function of *n* from a pattern.

Example 4:
Write an expression for the *n*th term.

(a) 3, 8, 13, 18, . . .

(b) 5, -15, 45, -135, . . .

(c) 1, 4, 9, 16, 25, . . .

(d) 4, 10, 28, 82, . . .

(e) $\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \ldots$

(f) $\ln 1,\ \ln 2,\ \ln 4,\ \ln 8, \ldots$

A **Series** is the sum of the terms in a sequence. Finite sequences and series have defined first and last terms, whereas infinite sequences and series continue indefinitely. A series is informally the result of adding any number of terms from a sequence together: $a_1 + a_2 + a_3 + \cdots$. A series can be written more succinctly by using the summation symbol sigma, $\sum .$, the Greek letter "S" for **Esum** (the "E" is both silent and not really there.)

For infinite series, we can look at the sequence **of partial sums**, that is, looking to see what the sums are doing as we add additional terms. In general, the *n*th partial sum of a series is denoted S_n. This can be explored on a calculator by adding sequential terms to the aggregate sum.

Example 5:
For both $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$, generate the sequence of partial sums $S_1,\ S_{2,}\ S_{3,}\ \ldots, S_n$, for each, then determine if the sequences converges or diverges. Do the results surprise you? Where else have we seen something like this before?

<u>Convergence and Divergence of a Series</u>

What does it mean for a series to converge? To diverge? Let's look at a couple series from a special family called **geometric series**.

Example 6:

Given the series $\sum_{n=1}^{\infty} \frac{3}{2^n} = \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \frac{3}{128} + \frac{3}{256} + \frac{3}{512} + \frac{3}{1024} + \cdots$,

find the first **ten** terms of the sequence of partial sums, and list them below, $S_1, S_2, S_3, \ldots, S_{10}$. Based on this sequence of partial sums, do you think the series converges? Diverges? To what? (HINT: first rewrite the rule of sequence so that it looks like an **exponential function** of *n*.)

Example 7:

Given the series $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n = \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \frac{81}{16} + \frac{243}{32} + \cdots$, find the first **<u>five</u>** terms of the sequence of partial sums, and list them below. Based on this sequence of partial sums, do you think the series converges? Diverges? To what?

We are now going to look at several families of infinite series and several tests that will help us determine whether they converge or diverge. For some that converge, we might be able to give the actual sum, or an interval in which we know the sum will be. For others, simply knowing that they converge will have to suffice.

<u>Geometric Series, nth Term Test for Divergence, and Telescoping Series</u>

<u>Geometric Series Test (GST)</u>

A geometric series is in the form $\sum_{n=0}^{\infty} a \cdot r^n$ or $\sum_{n=1}^{\infty} a \cdot r^{n-1}$, $a \neq 0$

The geometric series **<u>diverge</u>**s if $|r| \geq 1$.

If $|r| < 1$, the series **<u>converge</u>**s to the sum $S = \frac{a_1}{1-r}$.

Where a_1 is the first term, regardless of where *n* starts, and *r* is the common ration.

Example 8:

Using the GST, determine whether the following series converge or diverge. If the converge, find the sum.

(a) $\sum_{n=1}^{\infty} \frac{3}{2^n}$ (b) $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ (c) $\sum_{n=2}^{\infty} 3\left(-\frac{1}{2}\right)^n$

Example 9:

Determine if the following series converge or diverge. If convergent, find the sum.

(a) $\sum_{n=0}^{\infty} \left(\frac{2}{7}\right)^n$ (b) $\sum_{n=0}^{\infty} (-0.05)^n$ (c) $\sum_{n=5}^{\infty} 2\left(\frac{1}{2}\right)^n$ (d) $\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n}$ (e) $\sum_{n=1}^{\infty} \frac{3^n + 4}{2^n}$