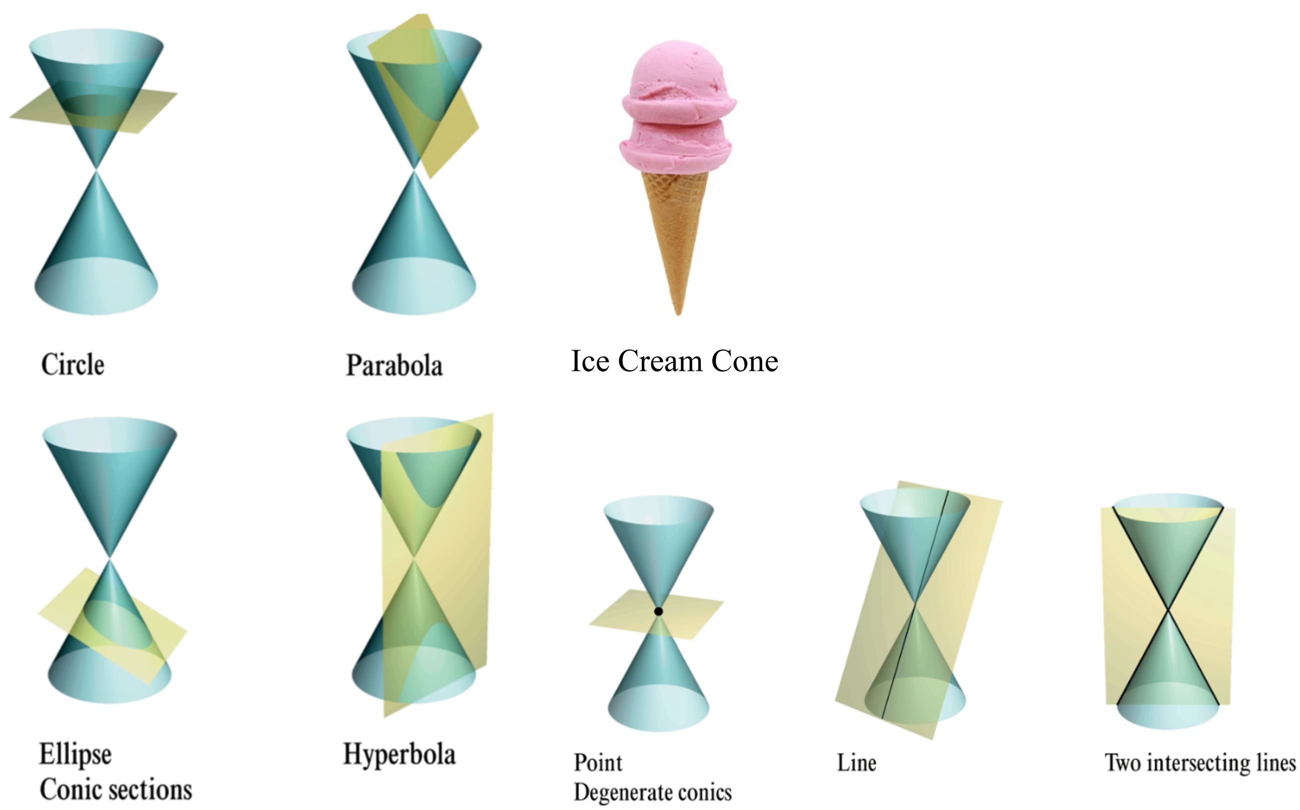
Chapter 9.1: Conic Sections—Circles

Ever heard of a cone? A right circular cone? How about a double-knapped right circular cone?? Are you craving ice cream now???

If we take planar cross sections of such a double-knapped right circular cone, we get four typical shapes, called conic sections, and three degenerate ones. The four typical Conic types are circles, ellipses, hyperbolas, and parabolas. The degenerate ones are a point, a line, and two intersecting lines. The diagram below shows them (and also a double scoop strawberry ice cream cone!)



In addition to this geometric representation of a conic section, we will study the algebra-based idea that these sections can be represented as a second-degree equation of two variables, as well as the locus (collection) definition stating that each of these conic sections satisfies a particular geometric condition.

We study conic sections because the practical applications of conic sections are numerous and varied. They each have very unique and practical reflective properties, and are used in physics, orbital mechanics, and optics, among others.

*NOTE: Apollonius of Perga lived in the third century before the common era and studied in Alexandria. His work on conic sections is massive and difficult. The names of the curves parabola, ellipse, and hyperbola are his. Apollonius' work not only influenced Ptolemy in his studies of planetary orbits, but Descartes and Fermat in the 17th century in their development of analytical geometry. Hypatia extended Apollonius' work and made notable commentaries on these Conics (which are lost to time), but the use of the Conics in astronomy and Apollonius' connection to Alexandria are arguments for Hypatia's involvement with commentaries on the Conics.

General Second-Degree Equation in Two Variables: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

Circle: $Ax^2 + Cy^2 + Dx + Ey + F = 0$, $A = C$

Ellipse: $Ax^2 + Cy^2 + Dx + Ey + F = 0$, $A \neq C$ and $AC > 0$ (A and C are same sign)

Hyperbola: $Ax^2 - Cy^2 + Dx + Ey + F = 0$ or $-Ax^2 + Cy^2 + Dx + Ey + F = 0$, where $A > 0, C > 0$

Parabola: $Ax^2 + Dx + Ey + F = 0$ or $Cy^2 + Dx + Ey + F = 0$

Example 1:
Identify each general equation of a Conic as a Circle, Ellipse, Hyperbola, or none of these

(a) $5x^2 + 7y^2 + 15x + 22y - 56 = 0$ (b) $8x^2 - 11y^2 + 78x + 900y - 644 = 0$

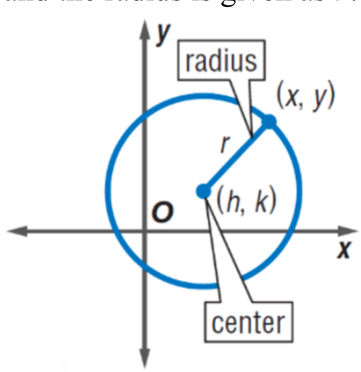
(c) $x^2 - 44x - 75y + 19 = 0$ (d) $x^2 + 6y^2 = 88$ (e) $5x^2 + 6y = -2y^2 + 9$

We will now focus our attention on the simplest of all of the Conics: the circle!

Locus Definition of a Circle

A circle is the set of all coplanar points equidistant from a fixed point, where the fixed point is the center of the circle and the distance of the sets of points from the center is the radius of the circle.

The center is given as the point (h, k) and the radius is given as r.



Recall the general equation of a circle is $Ax^2 + Cy^2 + Dx + Ey + F = 0$, where $A = C$.

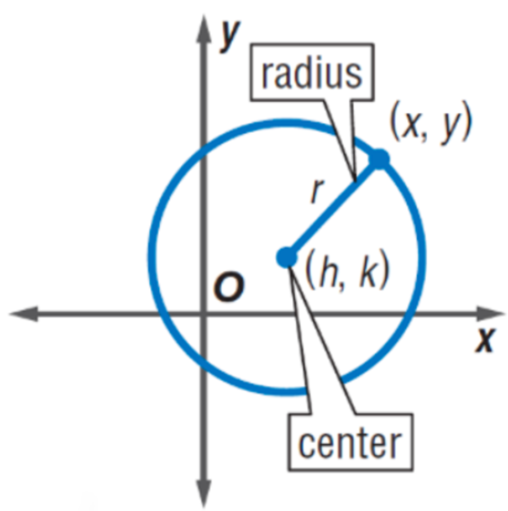
Geometrically, a circle is the cross-sectional shape when a cone is sliced *parallel to the base* of the cone. This Conic, as do all the others, has another form that is more useful for sketching a graph of the Conic. This form is called **standard form**. To algebraically go from general to standard form, we often have to *complete the square*.

Example 2:
Put the general equation of the circle $x^2 + y^2 - 6x - 4y - 12 = 0$ into standard form. Sketch the graph of the circle from your result. State the domain and range.

Standard form of an equation of a Circle

A circle whose center is at (h, k) with radius of r has a standard equation of

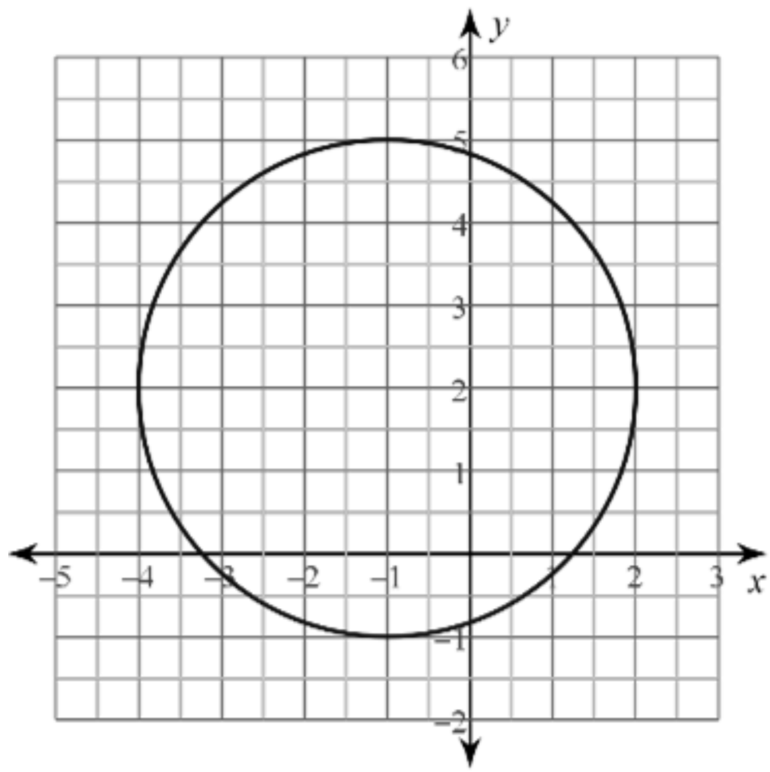
$$(x - h)^2 + (y - k)^2 = r^2$$



A circle centered at the origin $(0,0)$ has standard equation of $x^2 + y^2 = r^2$. Recognize this equation?

*The standard equation of the circle allows us to easily see at a glance both the center and the radius, the two things needed to sketch a graph!
**Notice for the circle centered at the origin, solving for positive y, $y = \sqrt{r^2 - x^2}$ gives the positive semicircle.

Example 3:
Find the standard equation of the given graph of the circle, then put the equation into general form.



Example 4:
Find the standard equation of the a circle whose center is $(2, -5)$ that passes through the point $(-7, -1)$. Sketch the graph and state the domain and range.

Example 5:
If a circle passes through the points $(2,0)$ and $(0,4)$ and if the center of the circle is on the *x*-axis, then what is the radius of the circle?

Example 6:
Find the standard equation of the a circle whose diameter has endpoints of $(18, -13)$ & $(4, -3)$. Sketch the graph and state the domain and range.

Example 7:
Given the circle whose equation is $x^2 + y^2 - 2x - 4y + 2 = 0$, determine if the point $(1,4)$ lies inside the circle, outside the circle, or on the circle.

Example 8:
Given the general equation $-20x + 4y^2 - 32y = -81 - 4x^2$, how can you tell this represents a circle? Find the center and radius of this circle by putting the equation into standard form. Sketch the graph and state the domain and range.

*Note: Each Conic has a value that measures the "roundness" of its shape. This is called the **eccentricity**, e, of the Conic. This e is not to be confused with Euler's number (context clues, people!). It turns out that for a circle, $e = 0$, which is easy to remember, because 0 itself is a circle, so a circle is perfectly round.